MATH1050 Examples: Complex numbers.

1. Let $\omega = \dfrac{1+i}{\sqrt{2}}$.

   (a) Write down the respective values of ω^2, ω^8, ω^{2016}.

   (b) Hence, or otherwise, find the value of $\left|\sum_{k=0}^{2017} \omega^k\right|^2$. Leave your answer in surd form.

2. Let ζ be a complex number with real part a and modulus r. Express $\zeta^m + \bar{\zeta}^m$ in terms of a, r alone for $m = 1, 2, 3, 4, 5, 6$.

3. Let k be a real number, and ζ be the complex number defined by $\zeta = (2+i)k^2 - 3(1+i)k - 2(1-i)$.

   (a) Express $\mathsf{Re}(\zeta)$ and $\mathsf{Im}(\zeta)$ in terms of k.

   (b) i. Suppose ζ is real. What are the possible values of k and ζ respectively? Justify your answer.

   ii. Suppose ζ is purely imaginary. What are the possible values of k and ζ respectively? Justify your answer.

   iii. Suppose $\mathsf{Re}(\zeta) = \mathsf{Im}(\zeta)$ and $\zeta \neq 0$. What are the possible values of k and ζ respectively? Justify your answer.

4. Let a, b, h, k be real numbers, with $h \neq 0$. Let ω be a complex number, with $|\omega| = 1$. Suppose $a + bi = \dfrac{h}{k+\omega}$.

   (a) Verify that $\omega = \dfrac{(h - ak) - bki}{a + bi}$.

   (b) Hence deduce that $(k^2 - 1)(a^2 + b^2) + h^2 - 2ahk = 0$.

5. Let z, w be complex numbers. Suppose $w \neq 0$.

   (a) Suppose $|z| = |z - w|$. Prove that $\mathsf{Re}\left(\dfrac{z}{w}\right) = \dfrac{1}{2}$.

   (b) Suppose $|z| = |z - w| = |w|$. Express z in terms of w.

6. Let α, β be complex numbers. Suppose $|\alpha| = |\beta| = |\alpha + \beta| = 1$.

   (a) Find the value of $(\alpha + \beta)\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$.

   (b) Hence, or otherwise, prove that $\alpha^3 = \beta^3$.

   Remark. Start by considering the expression $\alpha^3 - \beta^3$: can you factorize it?

7. Let α, β, γ be real numbers. Suppose $\alpha + \beta + \gamma = 2\pi$.

   Define λ, μ, ν by $\lambda = \cos(\alpha) + i\sin(\alpha)$, $\mu = \cos(\beta) + i\sin(\beta)$, $\nu = \cos(\gamma) + i\sin(\gamma)$ respectively.

   (a) Find the value of $\lambda\mu\nu$.

   (b) Prove that $\cos(\alpha) = \dfrac{1}{2}(\lambda + \dfrac{1}{\lambda})$ and $\cos(2\alpha) = \dfrac{1}{2}(\lambda^2 + \dfrac{1}{\lambda^2})$.

   (c) Hence, or otherwise, prove that $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 4\cos(\alpha)\cos(\beta)\cos(\gamma) - 1$.

8. Let z, w be complex numbers. Prove that $|1 - z\bar{w}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2)$.

9. Let a, b be real numbers. Suppose $a + bi = \dfrac{2+4i}{1-i}(b + i)$. Determine the values of a, b respectively.

10. Let a, b be real numbers. Consider the quadratic equation $z^2 + az + b = 0$ with unknown z. Suppose $z = 1 + i$ is a solution of this equation. Find the values of a, b.

11. Let a, b, c, d be real numbers, and $f(x)$ be the quadratic polynomial with complex coefficients given by $f(x) = x^2 + (a+bi)x + (c+di)$.

    Prove the statements below:

    (a) *$f(x)$ has a pair of distinct real root iff ($b = d = 0$ and $a^2 - 4c > 0$).*

    (b) *$f(x)$ has a pair of distinct complex roots which are conjugate to each other iff ($b = d = 0$ and $a^2 - 4c < 0$).*

    (c) *$f(x)$ has one real root and one non-real root iff ($b \neq 0$ and $d^2 - abd + b^2c = 0$).*

12. (a) Let ζ be a complex number. Suppose $\zeta^2 = \bar{\zeta}$.

    i. Prove that $|\zeta| = 0$ or $|\zeta| = 1$.

    ii. Hence, or otherwise, prove that $\zeta = 0$ or $\zeta = 1$ or $\zeta^2 + \zeta + 1 = 0$.

    (b) Solve the equation $z^2 = \bar{z}$ with unknown z in $\mathbb{C}$.

13. Solve the system of equations

$$\begin{cases} |1+z| &= |3-z| \\ z\bar{z} &= 4 \end{cases}$$

    with unknown z in $\mathbb{C}$.

    Remark. If you are not required to display any algebraic manipulation, can you write down the answer quickly by simply considering the geometry on the Argand plane?

14. *There is no need to give any justifications for your answers in this question.*

    Consider the curve C on the Argand plane defined by the equation $z\bar{z} - (2+3i)z - (2-3i)\bar{z} = 12$.

    Let $p = 4 - 5i$. Determine the points on the curve C on the Argand plane which are respectively of the maximum distance and of the minimum distance from the point p.

    Remark. Try to re-write the equation which defines C in such a way that you can identify C as some kind of curves familiar to you.

15. *There is no need to give any justifications for your answers in this question.*

    Let $w = 4 + 3i$, and C be the curve on the Argand plane defined by the equation $|z - 3i| = 2$. Determine the point(s) in C at minimum distance from w on the Argand plane.

16. *There is no need to give any justifications for your answers in this question.*

    Consider the infinite straight line ℓ defind by the equation $|z - (3+i)| = |z - (5+5i)|$ with unknown z in the complex numbers.

    (a) Re-express the equation in the form $\mathsf{Im}(z) = a\mathsf{Re}(z) + b$, in which a, b are real numbers. You have to give explicit values for a, b respectively.

    (b) What is the smallest possible value of $|z|$ if z lies on ℓ?

17. Express the complex numbers $-1-i$, $1-i$ in polar form. Hence simplify $\dfrac{-1-i}{(1-i)^5}$.

18. Let c be a real number, $z = 1 + i$, $w = (c+4) + (c-4)i$. Denote the angle between the line segment joining 0 and z and that joining 0 and w is θ. Suppose $\cos(\theta) = -\dfrac{3}{5}$. Find the possible value(s) of c.

19. Let p, q, r be three distinct non-zero complex numbers. Denote the points in the Argand plane represented by p, q, r by P, Q, R respectively. Denote the origin in the Argand plane by O. Suppose $OPQR$ is a rectangle in the Argand plane. Suppose $|p| = 2$, and $\arg(p) = \dfrac{\pi}{6}$. Suppose $|q - p| = 1$. Also suppose $\mathsf{Re}(r) < 0$.

    (Draw an appropriate diagram before you proceed any further.)

    (a) Express p, r in standard form.

(b) Hence find q. Express your answer in standard form.

20. (a) Let α, σ be complex numbers.

    i.$^{\diamondsuit}$ Suppose $\mathsf{Re}(\alpha) = a$, $\mathsf{Im}(\alpha) = b$, $\mathsf{Re}(\sigma) = s$, $\mathsf{Im}(\sigma) = t$.
    Prove that $\alpha = \sigma^2$ iff ($a = s^2 - t^2$ and $b = 2st$).

    ii. Now suppose $|\alpha| = 1$ and $\alpha = \sigma^2$.

       A. Prove that $|\sigma| = 1$.

       B. Prove that $|\alpha - \sigma| = |\sigma - 1|$.

       Remark. Can you give a geometric interpretation of these results on the Argand plane?

    (b) Find the square roots of each of the complex numbers below:

    i. i

    ii. $-i$

    iii. $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

    iv. $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

    v. $1 + 3i$

    vi. $1 + i$

21. Solve for all complex solutions of each of the equations below:

    (a) $z^2 + 4iz = 0$.

    (b) $z^2 + (3 - 2i)z - 6i = 0$.

    (c) $z^2 - 2iz + 1 = 0$.

    (d) $z^2 - 4z + (4 - i) = 0$.

    (e) $z^2 - (2 + 2i)z + (-4 + 2i) = 0$.

    (f) $z - 5i - \dfrac{6}{z} = 0$.

22. (a) Find the square roots of each of the complex numbers below:

    i. $4 + 3i$

    ii. $4 - 3i$

    (b) Hence, or otherwise, solve for all complex solutions of the equation $x^4 - 8x^2 + 25 = 0$.