MATH1050 Roots of polynomials with complex coefficients.

1. Here we take for granted everything that you have learnt about polynomials at schools, up to the point where *calculus* is involved.

   Every definition and every result will remain valid when we replace the word *real* by the word *complex* and replace the symbol $\mathbb{R}$ by the symbol $\mathbb{C}$ in every definition/result concerned with polynomials.

   In particular:—

   - When $f(z)$ is a polynomial with complex coefficients, we are allowed to 'substitute' a number, say, α, into its indeterminate z to obtain the number $f(\alpha)$.
   - When $f(z), g(z)$ are polynomials, we will say that $f(z)$ is divisible by $g(z)$, exactly when there is a polynomial $k(z)$ such that $f(z) = k(z)g(z)$.
   - **Remainder Theorem** holds:
     Suppose $f(z)$ is a polynomial with complex coefficients, and α is a complex number. Then there exists some unique polynomial $g(x)$ with complex coefficients and some unique complex number r, namely $r = f(\alpha)$, such that $f(z) = (z - \alpha)g(z) + r$ as polynomials.
   - **Factor Theorem** holds:
     Suppose $f(z)$ is a polynomial with complex coefficients, and α is a complex number.
     Then $f(z)$ is divisible by $z - \alpha$ iff $f(\alpha) = 0$.

2. **Definition. (Roots of polynomials.)**

   Let $f(z)$ be a polynomial with complex coefficients. Let α be a complex number.

   We say α is a **root of the polynomial** $f(z)$ **in** $\mathbb{C}$ *if $f(\alpha) = 0$.*

3. **Theorem (1).**

   Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

   The polynomial $z^n - 1$ with indeterminate z is completely factorized as

$$z^n - 1 = (z-1)(z-\omega_n)(z-{\omega_n}^2) \cdot \ldots \cdot (z-{\omega_n}^{n-1}).$$

4. **Proof of Theorem (1).** Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

   Write $f(z) = z^n - 1$.

   (a) By definition, each of $1, \omega_n, {\omega_n}^2, \cdots, {\omega_n}^{n-1}$ is a root of the polynomial $f(z)$.

   (b) In particular $f(1) = 0$.
   Then by Factor Theorem, there exists some polynomial $f_1(z)$ such that $f(z) = (z-1)f_1(z)$ as polynomials.

   (c) We have $0 = f(\omega_n) = (\omega_n - 1)f_1(\omega_n)$.
   Since $\omega_n \neq 1$, we have $f_1(\omega_n) = 0$.
   Then by Factor Theorem, there exists some polynomial $f_2(z)$ such that $f_1(z) = (z-\omega_n)f_2(z)$ as polynomials.
   So $f(z) = (z-1)(z-\omega_n)f_2(z)$ as polynomials.

   (d) We have $0 = f({\omega_n}^2) = ({\omega_n}^2 - 1)({\omega_n}^2 - \omega_n)f_2(\omega_2)$.
   Since ${\omega_n}^2$ is distinct from each of $1, \omega_n$, we have $f_2({\omega_n}^2) = 0$.
   Then by Factor Theorem, there exists some polynomial $f_3(z)$ such that $f_2(z) = (z - {\omega_n}^2)f_3(z)$ as polynomials.
   Then $f(z) = (z-1)(z-\omega_n)(z-{\omega_n}^2)f_3(z)$ as polynomials.

   (e) Repeating this argument, we deduce that there exists some polynomial $g(z)$ such that

$$f(z) = (z-1)(z-\omega_n)(z-{\omega_n}^2) \cdot \ldots \cdot (z-{\omega_n}^{n-1})g(z)$$

   as polynomials.

Note that $(z-1)(z-\omega_n)(z-{\omega_n}^2)\cdot\ldots\cdot(z-{\omega_n}^{n-1})$ is a degree-n polynomial with leading coefficient 1.
$f(z)$ is also a degree-n polynomial with leading coefficient 1.
Then $g(z)$ is the constant polynomial 1.
Therefore $f(z)=(z-1)(z-\omega_n)(z-{\omega_n}^2)\cdot\ldots\cdot(z-{\omega_n}^{n-1})$ as polynomials.

5. **Theorem (2).**

Let n be a positive integer. Write $\theta_n=\dfrac{2\pi}{n}$. Define $\omega_n=\cos(\theta_n)+i\sin(\theta_n)$.

Let v be a non-zero complex number, and ζ be an n-th root of v.

The polynomial z^n-v with indeterminate z is completely factorized as

$$z^n-v=(z-\zeta)(z-\zeta\omega_n)(z-\zeta{\omega_n}^2)\cdot\ldots\cdot(z-\zeta{\omega_n}^{n-1}).$$

Proof of Theorem (2). Exercise. (Imitate the argument for Theorem ().)

Remark. Suppose φ is an argument for v. Take $\zeta_0=\sqrt[n]{|v|}(\cos(\dfrac{\varphi}{n})+i\sin(\dfrac{\varphi}{n}))$. Then ζ_0 is an n-th root of v. Therefore

$$z^n-v=(z-\zeta_0)(z-\zeta_0\omega_n)(z-\zeta_0{\omega_n}^2)\cdot\ldots\cdot(z-\zeta_0{\omega_n}^{n-1})$$

as polynomials.

6. **Theorem (3).**

Let n be a positive integer. Write $\theta_n=\dfrac{2\pi}{n}$. Define $\omega_n=\cos(\theta_n)+i\sin(\theta_n)$.

Suppose s,t are complex numbers.

Then the equality

$$s^n-t^n=(s-t)(s-t\omega_n)(s-t{\omega_n}^2)\cdot\ldots\cdot(s-t{\omega_n}^{n-1})$$

holds.

Proof of Theorem (3). Exercise. (Apply Theorem (2).)

7. Using the same argument as in Theorem (1), we can prove Theorem (4).

Theorem (4).

Let $f(z)$ be a non-constant polynomial with complex coefficients. Suppose the degree of $f(z)$ is n.

Then $f(z)$ has at most n distinct roots in $\mathbb{C}$.

Polynomials with real coefficients are automatically polynomial with complex coefficients. Hence Theorem (4) trivially gives rise to the corollary below:

Corollary to Theorem (4).

Let $f(x)$ be a non-constant polynomial with real coefficients. Suppose the degree of $f(x)$ is n.

Then $f(x)$ has at most n distinct roots in $\mathbb{C}$. In particular, $u(x)$ has at most n distinct roots in $\mathbb{R}$.

8. **Proof of Theorem (4).** Exercise in proof-by-contradiction argument. Fill in the missing steps in the outline below:

Suppose $f(z)$ is a non-constant polynomial with complex coefficients. Suppose the degree of $f(z)$ is n. By assumption $n\geq 1$.
Suppose it were true that $f(z)$ had more than n distinct roots in $\mathbb{C}$. Suppose $\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_n$ were $n+1$ distinct roots of $f(z)$ in $\mathbb{C}$.
By assumption, $f(\alpha_j)=0$ for each $j=0,1,2,\cdots,n$.
By applying the Factor Theorem, we deduce that there would exist some polynomial $g(z)$ such that $f(z)=(z-\alpha_0)(z-\alpha_1)(z-\alpha_2)\cdot\ldots\cdot(z-\alpha_n)g(z)$ as polynomials. (Fill in the missing steps.)
Then $f(z)$ would be of degree at least $n+1$. (Fill in the missing steps.)
However, by assumption the degree of $f(z)$ is n. Contradiction arises.
Hence $f(z)$ has at most n distinct roots in $\mathbb{C}$ in the first place.

9. We know that every quadratic polynomial with complex coefficients has exactly two (not necessarily) distinct roots in $\mathbb{C}$, and completely factorize into a product of linear polynomials. (Refer to the handout *Polar form.*)

   This result can be generalized to non-constant polynomials of arbitrary degrees.

   Theorem (5). (Factorization of polynomials with complex coefficients into 'linear factors'.)

   Let $f(z)$ be a non-constant polynomial with complex coefficients. Suppose the degree of $f(z)$ is n. Suppose the leading coefficient of $f(z)$ is a_n.

   Then there exist some n complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, not necessarily distinct, such that

   $$f(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdot ... \cdot (z - \alpha_n) \qquad \text{as polynomials.}$$

   These n numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all the roots of $f(z)$ in $\mathbb{C}$.

10. The result in Theorem (1) and Theorem (2) are manifestations of Theorem (5) in a special case.

    The proof of Theorem (5) relies on a non-trivial result, called the **Fundamental Theorem of Algebra**, first proved by Gauss. Here we take the validity of this result for granted. (Its proof is beyond the scope of this course.)

    Fundamental Theorem of Algebra.

    Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$.

11. **Proof of Theorem (5).** Exercise in mathematical induction. Fill in the steps in the roughwork below:

    For each positive integer n, denote by $P(n)$ the proposition below:

    'Suppose $f(z)$ is a polynomial of degree n. Suppose the leading coefficient of $f(z)$ is a_n. Then there exist some n complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, not necessarily distinct, such that

    $$f(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdot ... \cdot (z - \alpha_n)$$

    as polynomials.'

    $P(1)$ is trivial true.

    In the inductive argument, apply the Fundamental Theorem of Algebra and the Factor Theorem to an arbitrary degree-$(k+1)$ polynomial so as ensure that it is factorized as a product of a linear polynomial and a degree-k polynomial. Then apply the induction hypothesis on that degree-k polynomial.

12. **Theorem (6). (Vieta's Theorem, relating roots and coefficients of polynomials.)**

    Let $f(z)$ be a polynomial with complex coefficients, of degree $n \geq 1$, with its k-th coefficient being a_k for each $k = 0, 1, 2 \cdots, n$.

    Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all the n roots of $f(z)$ in $\mathbb{C}$.

    Then

    $$\begin{cases} -\dfrac{a_{n-1}}{a_n} &= \displaystyle\sum_{k=1}^{n} \alpha_k, \\ \dfrac{a_{n-2}}{a_n} &= \displaystyle\sum_{1 \leq j_1 < j_2 \leq n} \alpha_{j_1}\alpha_{j_2}, \\ -\dfrac{a_{n-3}}{a_n} &= \displaystyle\sum_{1 \leq j_1 < j_2 < j_3 \leq n} \alpha_{j_1}\alpha_{j_2}\alpha_{j_3}, \\ & \vdots \\ (-1)^n \cdot \dfrac{a_0}{a_n} &= \alpha_1\alpha_2 \cdot ... \cdot \alpha_n \end{cases}$$

    Proof of Theorem (6). This is a tedious exercise in 'comparing coefficients' for the two sides of the equality

    $$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = a_n(z - \alpha_1)(z - \alpha_2) \cdot ... \cdot (z - \alpha_n) \quad \text{as polynomials,}$$

    Theorem (5) has been used in guaranteeing the availability of such an equality.

13. **Digression on Theorem (6).**

Below are some special cases of Theorem (6), for 'polynomials of low degrees':

(a) Suppose $n = 2$. Then Theorem (6) gives $-\dfrac{a_1}{a_2} = \alpha_1 + \alpha_2$, $\dfrac{a_0}{a_2} = \alpha_1\alpha_2$.

These equalities relate the coefficients of the quadratic polynomial $f(z)$ with its sum of roots and its product of roots. You have learnt them in school maths.

Out of these equalities we can obtain a 'formula' for the individual α_j'2 in terms of the a_k's, through the application of addition, subtraction, multiplication, division and extraction of square roots.

This is the 'quadratic formula' for the solutions of the equation $a_2z^2 + a_1z + a_0 = 0$.

(b) Suppose $n = 3$. Then Theorem (6) gives

$$-\frac{a_2}{a_3} = \alpha_1 + \alpha_2 + \alpha_3, \quad \frac{a_1}{a_3} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1, \quad -\frac{a_0}{a_3} = \alpha_1\alpha_2\alpha_3.$$

After some hard work, we can obtain a 'formula' for the individual α_j's in terms of the $a_k's$, through the application of addition, subtraction, multiplication, division, extraction of square roots and cubic roots (and with a little help from the cubic roots of unity).

This formula is known as the 'cubic formula' for the solution of the cubic equation $a_3z^3+a_2z^2+a_1z+a_0 = 0$.

(c) Suppose $n = 4$. Then Theorem (6) gives

$$\begin{cases} -\dfrac{a_3}{a_4} &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ \dfrac{a_2}{a_4} &= \alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_2\alpha_3, \\ -\dfrac{a_1}{a_4} &= \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 \\ \dfrac{a_0}{a_4} &= \alpha_1\alpha_2\alpha_3\alpha_4. \end{cases}$$

After some hard work, we can obtain a 'formula' for the individual α_j's in terms of the $a_k's$, through tha application of addition, subtraction, multiplication, division, extraction of square roots, cubic roots and quartic roots (and with a little help from the cubic roots of unity and the quartic roots of unity).

This formula is known as the 'quartic formula' for the solution of the quartic equation $a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 = 0$.

(d) How about the situation for '$n \geq 5$'?

We can certainly write down n equalities relating the α_j's and the a_k's. However, as discovered by Galois and Abel in the early nineteenth century, we do not have 'formulae' for the individual α_j's in terms of the $a_k's$, through the application of addition, subtraction, multiplication, division, extraction of square roots, cubic roots, ..., n-th roots alone.

Why this happens will be explained in the course *Fields and Galois Theory*.

14. From now on we focus on polynomials with real coefficients.

Theorem (7). ('Pairing-up' of complex roots for polynomials with real coefficients.)

Let $f(z)$ be a polynomial with real coefficients. Let α be a complex number.

Suppose α is a root of $f(z)$ in $\mathbb{C}$.

Then $\overline{\alpha}$ is also a root of $f(z)$ in $\mathbb{C}$.

15. **Proof of Theorem (7).**

Let $f(z)$ be a polynomial with real coefficients, say, of degree n. For each $k \in \mathbb{N}$, denote by a_k the k-th coefficient of $f(z)$.

Let α be a complex number. Suppose α is a root of $f(z)$ in $\mathbb{C}$. Then $0 = f(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$.

By assumption $a_k \in \mathbb{R}$ for each $k = 0, 1, 2, \cdots, n$.

Then

$$\begin{aligned} 0 &= \overline{f(\alpha)} = \overline{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n} \\ &= \overline{a_0} + \overline{a_1\alpha} + \overline{a_2\alpha^2} + \cdots + \overline{a_n\alpha^n} \\ &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^2 + \cdots + a_n\overline{\alpha}^n = f(\overline{\alpha}). \end{aligned}$$

Therefore $\overline{\alpha}$ is also a root of $f(z)$ in $\mathbb{C}$.

16. **Theorem (8).**

Let $f(z)$ be a polynomial with real coefficients. Let α be a complex number.

Suppose α is a non-real root of $f(z)$ in $\mathbb{C}$.

Then $f(z)$ is divisible by the quadratic polynomial with real coefficients $z^2 - 2\mathsf{Re}(\alpha)z + |\alpha|^2$, which has a negative discriminant.

Proof of Theorem (8). Exercise. (Apply the Factor Theorem to show that there exists some polynomial $g(z)$ such that $f(z) = (z - \alpha)(z - \overline{\alpha})g(z)$ as polynomials. Et cetera.)

17. With the help of Theorem (8), we can deduce the result below.

Theorem (9). (Factorization of polynomials with real coefficients into linear or quadratic factors.)

Suppose $f(z)$ is a non-constant polynomial with real coefficients.

Then $f(z)$ factorizes as a product of linear polynomials with real coefficients and quadratic polynomials with real coefficients of negative discriminant.

Proof of Theorem (9). Omitted. (The argument is tedious but not technically difficult.)

18. **Illustrations on the content of Theorem (9).**

(a) $x^3 - 1 = (x - 1)(x^2 + x + 1)$ as polynomials.

(b) $x^4 - 4 = (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)$ as polynomials.

(c) $x^4 + 2x^2 + 1 = (x^2 + 1)^2$ as polynomials.

(d) $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$ as polynomials.

(e) $x^5 - 1 = (x - 1)(x^2 - \dfrac{\sqrt{5}-1}{2}x + 1)(x^2 + \dfrac{\sqrt{5}+1}{2}x + 1)$ as polynomials.

(f) $x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$ as polynomials.

Some of these examples are in fact special cases described by Theorem (10).

19. **Theorem (10).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) *Suppose n is odd. Then the polynomial $z^n - 1$ is factorized as a product of polynomials with real coefficients given by*

$$z^n - 1 = (z - 1)(z^2 - 2z\cos(\theta_n) + 1)(z^2 - 2z\cos(2\theta_n) + 1) \cdot \ldots \cdot (z^2 - 2z\cos(\frac{n-1}{2}\theta_n) + 1).$$

(b) *Suppose n is even. Then the polynomial $z^n - 1$ is factorized as a product of polynomials with real coefficients given by*

$$z^n - 1 = (z - 1)(z + 1)(z^2 - 2z\cos(\theta_n) + 1)(z^2 - 2z\cos(2\theta_n) + 1) \cdot \ldots \cdot (z^2 - 2z\cos(\frac{n-2}{2}\theta_n) + 1).$$