1. Recall the notions of *real part*, *imaginary part*, *conjugate* and *modulus*, introduced in the handout *Basic algebraic results on complex numbers 'beyond school mathematics'*:

   Let z be a complex number. Denote the real part and the imaginary part of z by $\mathsf{Re}(z)$, $\mathsf{Im}(z)$ respectively. (So $z = \mathsf{Re}(z) + i\mathsf{Im}(z)$.)

   (a) *The* **complex conjugate** *of z is defined to be the complex number $\mathsf{Re}(z) - i\mathsf{Im}(z)$. It is denoted by $\overline{z}$.*

   (b) *The* **modulus** *of z is defined to be the non-negative real number $\sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}$. It is denoted by $|z|$.*

   Also recall the notions of *polar form* and *argument* from the handout *Polar form*:

   Let z be a complex number.
   When we write $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate real number θ), we say we are presenting z in its **polar form**.
   When $z \neq 0$, such a number θ is called an **argument** *for z.*

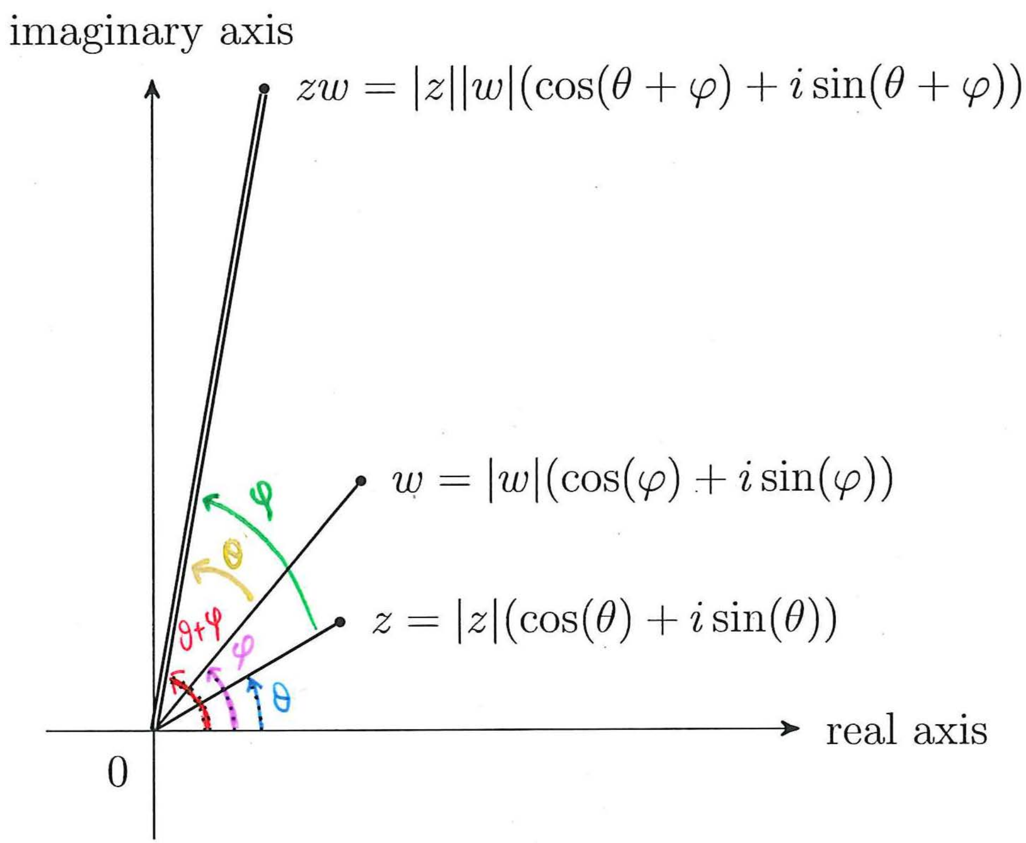
Further recall how the polar form for the product of two complex numbers is related to the polar forms of the two complex numbers concerned:

Suppose z, w are non-zero complex numbers, with arguments θ, φ respectively. Then:

(a) $zw = |z||w|(\cos(\theta + \varphi) + i\sin(\theta + \varphi))$.

(b) *The modulus of zw is $|z||w|$.*

(c) *$\theta + \varphi$ is an argument for zw.*



2. **Lemma (1). (Special case of De Moivre's Theorem.)**

Suppose θ is a real number.

Then for any $n \in \mathbb{N}\backslash\{0\}$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

Proof of Lemma (1).

Suppose θ is a real number.

- For any $n \in \mathbb{N}\backslash\{0\}$, denote by $P(n)$ the proposition $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.
- $(\cos(\theta) + i\sin(\theta))^1 = \cos(1 \cdot \theta) + i\sin(1 \cdot \theta)$. Then $P(1)$ is true.
- Let $k \in \mathbb{N}\backslash\{0\}$. Suppose $P(k)$ is true.
  Then $(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$.
  We prove that $P(k+1)$ is true:
  $$\begin{aligned} & (\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + i\sin(\theta))^k(\cos(\theta) + i\sin(\theta)) \\ = \ & (\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta)) \\ = \ & (\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta)) \\ = \ & \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos((k+1)\theta) + i\sin((k+1)\theta) \end{aligned}$$
  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

3. **De Moivre's Theorem.**

Suppose θ is a real number.

Then for any $m \in \mathbb{Z}$, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

Proof.

Suppose θ is a real number. Pick any $m \in \mathbb{Z}$. We have $m = 0$ or $m > 0$ or $m < 0$.

- (Case 1). Suppose $m = 0$. Then

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^m &= (\cos(\theta) + i\sin(\theta))^0 \\ &= 1 = (\cos(0\cdot\theta) + i\sin(0\cdot\theta)) = \cos(m\theta) + i\sin(m\theta).\end{aligned}$$

- (Case 2). Suppose $m > 0$.

  By Lemma (1), we have $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

- (Case 3). Suppose $m < 0$. Define $n = -m$. Then $n \in \mathbb{N}\backslash\{0\}$. Therefore

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^m = \frac{1}{(\cos(\theta) + i\sin(\theta))^n} &= \frac{1}{\cos(n\theta) + i\sin(n\theta)} \\ &= \cos(n\theta) - i\sin(n\theta) \\ &= \cos(m\theta) + i\sin(m\theta).\end{aligned}$$

Hence in any case, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

4. **Definition. (Roots of unity.)**

Suppose ζ is a complex number and n is a positive integer.

Then ζ is called an **n-th root of unity** *if $\zeta^n = 1$.*

Remark. ζ is an n-th root of unity iff ζ is a root of the polynomial $z^n - 1$ in the complex numbers.)

According to Theorem (3), stated below, we can pinpoint, for each positive integer n, exactly which numbers are n-th roots of unity.

5. **Lemma (2).**

Suppose n is a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Then ω_n is an n-th root of unity.

Proof of Lemma (2).

Suppose n is a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

By De Moivre's Theorem, we have

$$(\omega_n)^n = (\cos(n\theta_n) + i\sin(n\theta_n)) = \cos(2\pi) + i\sin(2\pi) = 1.$$

6. **Theorem (3).**

Suppose n is a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$. Then the n-th roots of unity are the n complex numbers of modulus 1, given by

$$1,\ \omega_n,\ {\omega_n}^2,\ \dots,\ {\omega_n}^{n-1}.$$

Remark. This is what the conclusion part of Theorem (3) is saying:—

- *Each of the n numbers $1, \omega_n, {\omega_n}^2, \dots, {\omega_n}^{n-1}$ is an n-th root of unity,* and
- *if ζ is an n-th root of unity, then ζ is amongst the n numbers $1, \omega_n, {\omega_n}^2, \dots, {\omega_n}^{n-1}$.*

Tacit assumption needed in the argument for Theorem (2).

A tacit assumption, known as **Division Algorithm for integers**, is used in the argument. It reads:

Let $u, v \in \mathbb{Z}$. Suppose $v > 0$.
Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq r < v$.

7. **Visualization of the n-th roots of unity on the Argand plane.**

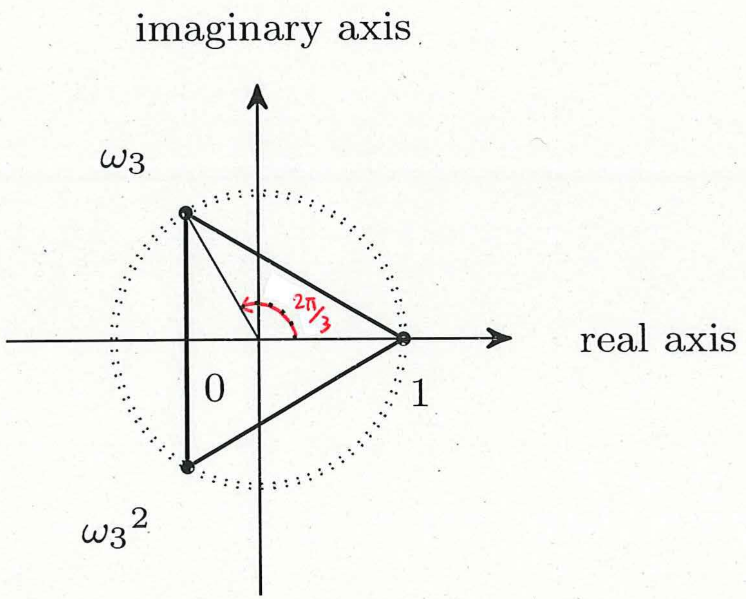
For each positive integer n, the n-th roots of unity are the

n vertices of the regular n-sided polygon
inscribed in the unit circle with centre 0

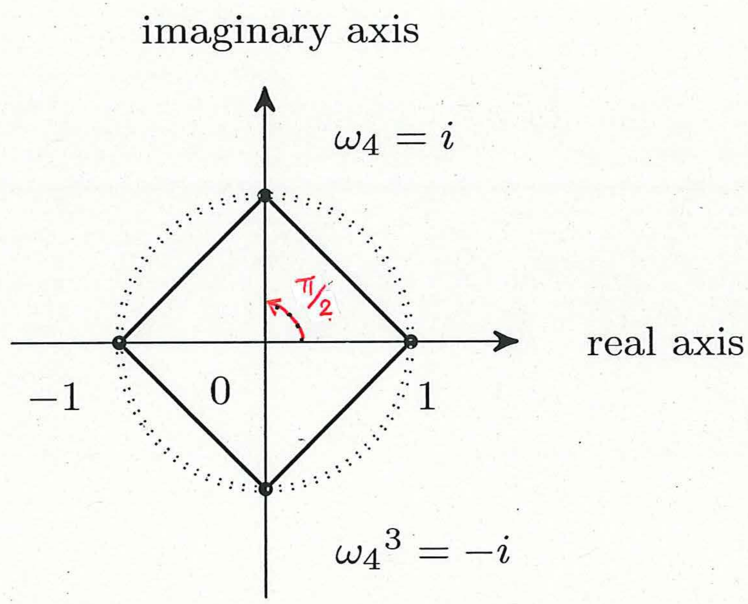
in the Argand plane,

with one vertex at the point 1.

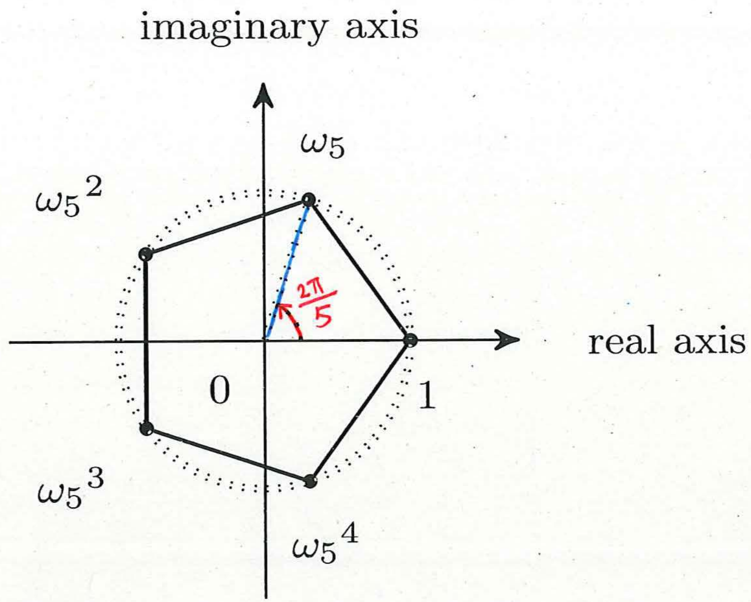
$n = 3$:



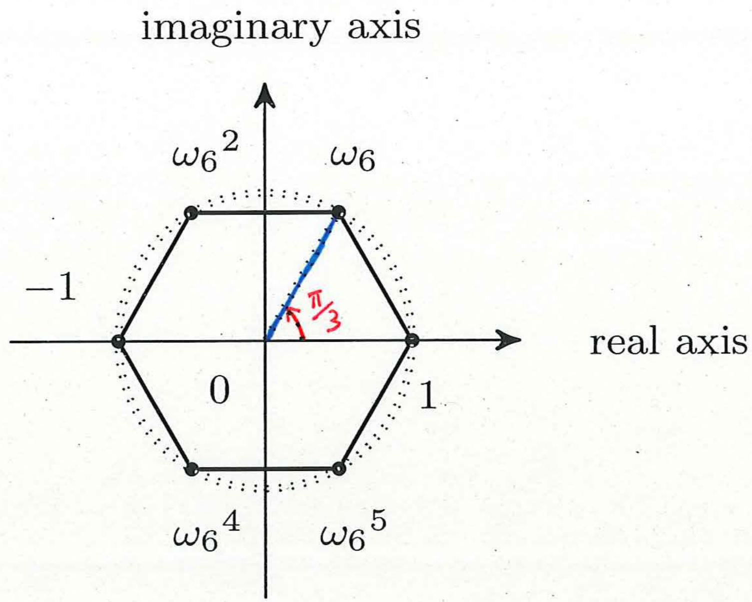
$n = 4$:



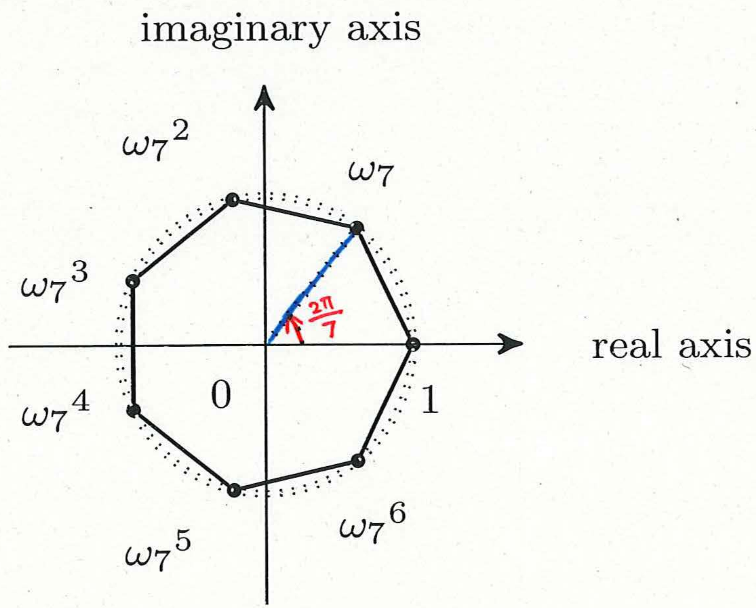
$n = 5$:


imaginary axis
ω_5
${\omega_5}^2$
$\frac{2\pi}{5}$
real axis
0
1
${\omega_5}^3$
${\omega_5}^4$

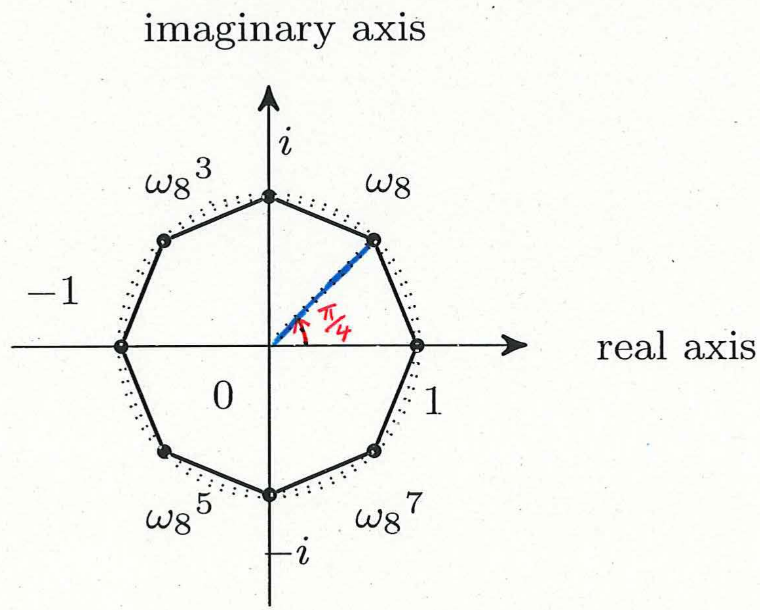
$n = 6$:


imaginary axis
${\omega_6}^2$
ω_6
-1
$\pi/3$
real axis
0
1
${\omega_6}^4$
${\omega_6}^5$

$n = 7$:


imaginary axis
${\omega_7}^2$
ω_7
${\omega_7}^3$
$\frac{2\pi}{7}$
real axis
${\omega_7}^4$
0
1
${\omega_7}^6$
${\omega_7}^5$

$n = 8$:


imaginary axis
i
${\omega_8}^3$
ω_8
-1
$\pi/4$
real axis
0
1
${\omega_8}^5$
${\omega_8}^7$
$-i$

Tacitly assumed result to be applied at (*):
Let $u, v \in \mathbb{Z}$. Suppose $v > 0$. Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq r < |v|$.

8. **Proof of Theorem (3).**

Suppose n is a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) For each $k = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = ({\omega_n}^n)^k = 1^k = 1$ by Lemma (2).

(b) Let ζ be a complex number. Suppose ζ is an n-th root of unity. Then $\zeta^n = 1$.

[We want to deduce that $\zeta = {\omega_n}^r$ for some $r \in [\![0, n-1]\!]$.]

We have $|\zeta|^n = |\zeta^n| = 1$. Then $|\zeta| = 1$.

ζ has an argument, say, φ. Therefore $\zeta = \cos(\varphi) + i\sin(\varphi)$.

[Ask: what more can we say about φ?]

By De Moivre's Theorem, $1 = \zeta^n = (\cos(\varphi) + i\sin(\varphi))^n = \cos(n\varphi) + i\sin(n\varphi)$.

Then $\cos(n\varphi) = 1$ and $\sin(n\varphi) = 0$.

Therefore there exists some $m \in \mathbb{Z}$ such that $n\varphi = 2m\pi$.

Now $\varphi = \dfrac{m}{n} \cdot 2\pi = m\theta_n$

(*) ⇝ By Division Algorithm, there exist some $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r < n$.

Then $\varphi = m\theta_n = (qn + r)\theta_n = qn\theta_n + r\theta_n = 2q\pi + r\theta_n$

Therefore $\zeta = \cos(\varphi) + i\sin(\varphi) = \cos(r\theta_n) + i\sin(r\theta_n) = {\omega_n}^r$. □

9. **Definition. (n-th roots of a complex number.)**

Suppose n is a positive integer, and w, ζ are complex numbers.

Then we say ζ is an **n-th root** *of w if $\zeta^n = w$.*

Remark.

ζ is an n-th root of w iff ζ is a root of the polynomial $z^n - w$ in the complex numbers.

Warning.

As we shall see from Theorem (5), whenever w is a non-zero complex number, there will be n complex numbers which are n-th roots of w.

It is not apparent whether any should be privileged over any other.

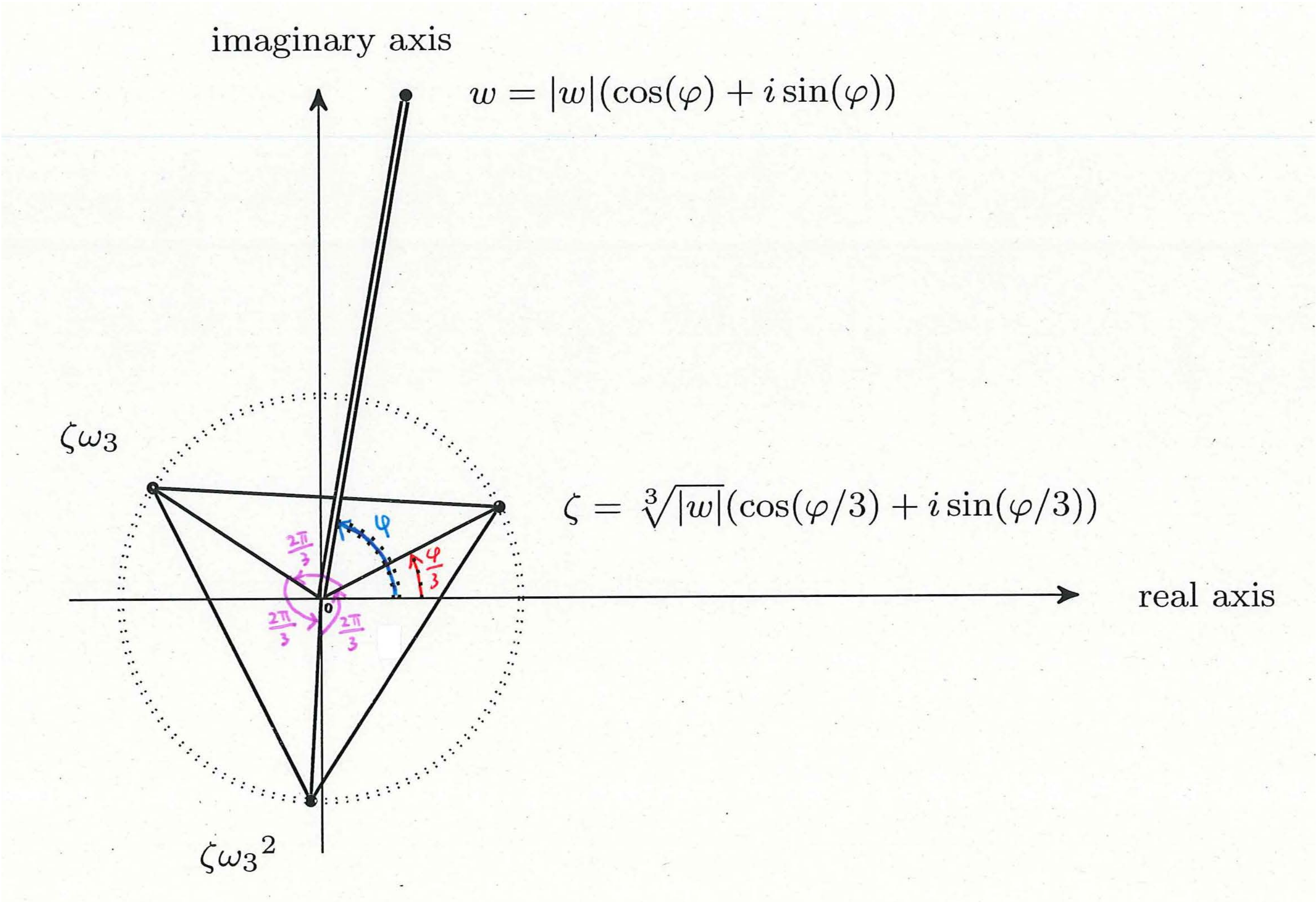
For this reason:—

- Never write '*the* n-th root of the complex number w' unless you are referring to a specific n-th root of the complex number w that you have already pinpointed.
- Never write '$\sqrt[n]{w}$' unless w is a non-negative real number.
  (When w is a non-negative real number, we 'privilege' its non-negative n-th root over all other n-th roots of w.)

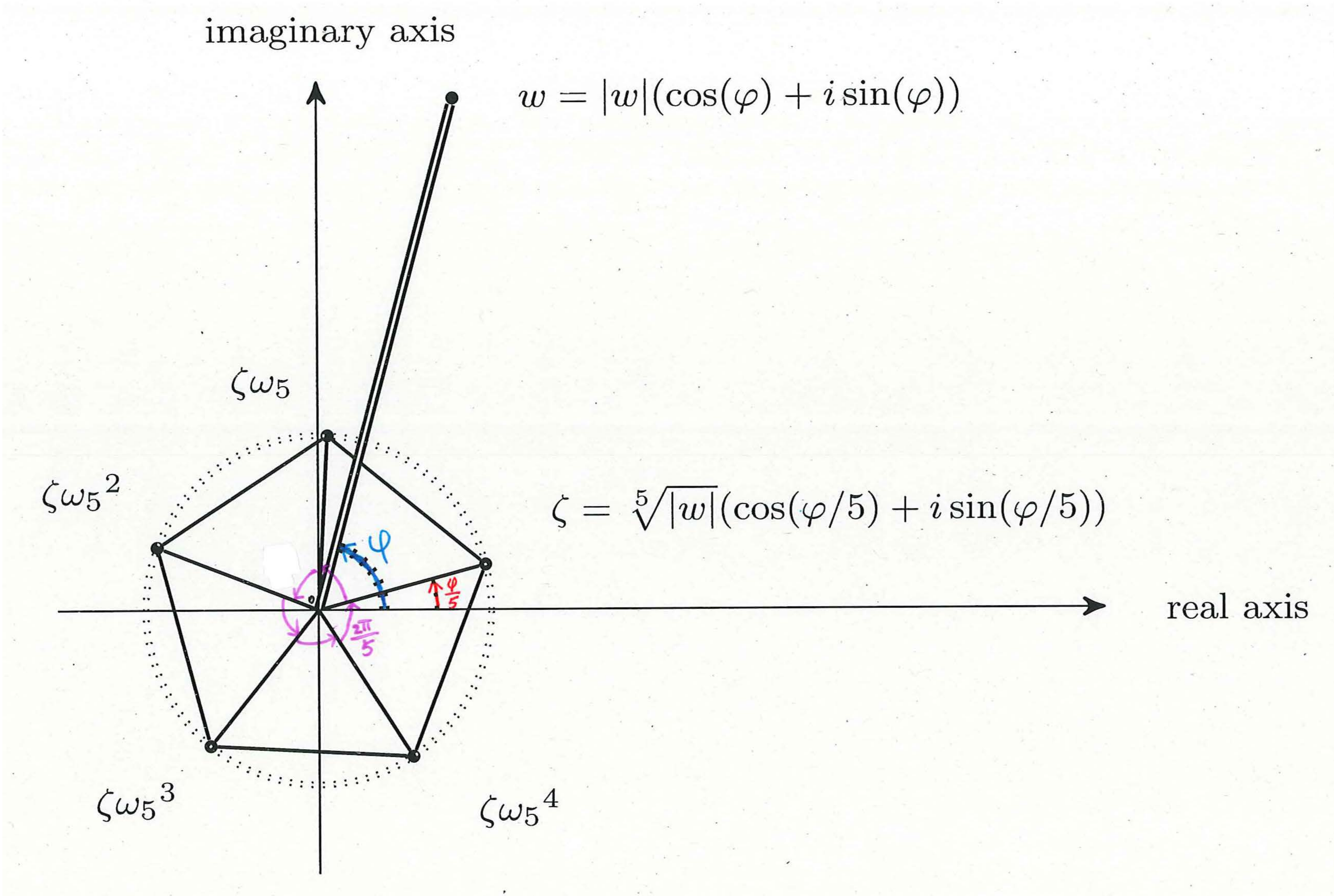
10. **Visualization of n-th roots of a complex number in the Argand plane.**

Suppose w is a non-zero complex number, with an argument φ. The n-th roots of a non-zero complex number w are the n vertices of the regular n-sided polygon inscribed in the circle with centre 0 and radius $\sqrt[n]{|w|}$ in the Argand plane, with one vertex at the point $\zeta = \sqrt[n]{|w|}(\cos(\varphi/n) + i\sin(\varphi/n))$.

- Cubic roots:



- Quintic roots:



11. **Theorem (4).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number.

Suppose ζ is an n-th root of w.

Then the n-th roots of w are the n complex numbers given by $\zeta, \zeta\omega_n, \zeta{\omega_n}^2, \cdots, \zeta{\omega_n}^{n-1}$.

12. Applying De Moivre's Theorem and Theorem (4), we can deduce the result below.

Theorem (5).

Let n be a positive integer.

Let w be a non-zero complex number. Suppose φ is an argument for w.

Define $\zeta_0 = \sqrt[n]{|w|}(\cos(\varphi/n) + i\sin(\varphi/n))$.

Then the n-th root of w are given by $\zeta_0, \zeta_0\omega_n, \zeta_0{\omega_n}^2, \cdots, \zeta_0{\omega_n}^{n-1}$.

13. **Proof of Theorem (4).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

- We have $\zeta^n = w$.

  For each $n = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = 1$.

  Then $(\zeta {\omega_n}^k)^n = \zeta^n ({\omega_n}^n)^k = 1 \cdot 1^k = 1$.
- Let ρ be a complex number. Suppose ρ is an n-th root of w.

  Then $\rho^n = w$.

  We have $\left(\dfrac{\rho}{\zeta}\right)^n = \dfrac{\rho^n}{\zeta^n} = \dfrac{w}{w} = 1$.

  Then $\dfrac{\rho}{\zeta}$ is an n-th root of unity.

  Therefore there exists some $r = 0, 1, 2, \cdots, n-1$ such that $\dfrac{\rho}{\zeta} = {\omega_n}^r$.

  For the same r, we have $\rho = \zeta {\omega_n}^r$.