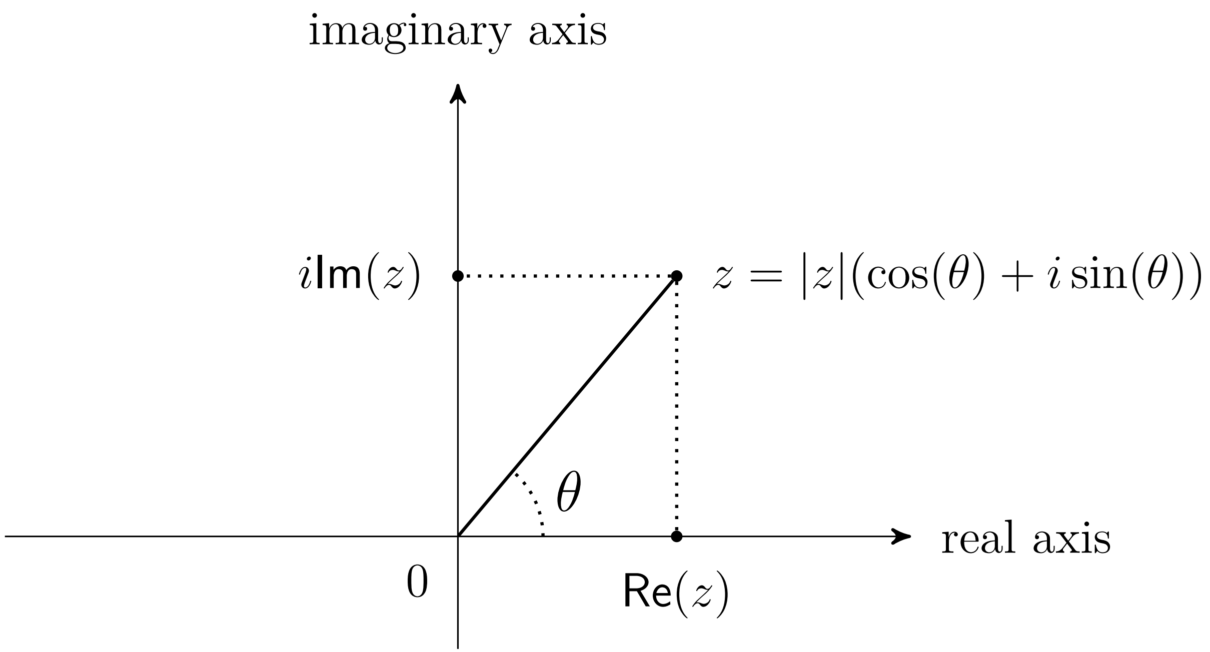
1. Whatever you have learnt (and tacitly taken for granted) about polar coordinates in coordinate geometry in school maths can be adapted to complex numbers through Theorem (1).

Theorem (1).

Suppose z is a complex number.

Then there exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.

Pictorial visualization of the content of Theorem (1).



Theorem (1).

Suppose z is a complex number.

Then there exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.

Remark.

(a) When we write

$$z = \mathsf{Re}(z) + i\mathsf{Im}(z),$$

we say we are presenting z in its **standard form**.

(b) When we write

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

(for some appropriate θ), we say we are presenting z in its **polar form**.

2. **Proof of Theorem (1).**

Starting from '$z = \mathsf{Re}(z) + i\,\mathsf{Im}(z)$', we have $|z| = \sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}$.

But can we name a θ satisfying $\begin{cases} \mathsf{Re}(z) = |z|\cos(\theta) \\ \mathsf{Im}(z) = |z|\sin(\theta) \end{cases}$?

Suppose z be a complex number.

Then we have $z = 0$ or ($z \neq 0$ and $\mathsf{Re}(z) \geq 0$) or ($z \neq 0$ and $\mathsf{Re}(z) < 0$).

- (Case 1). Suppose $z = 0$.

  Then (it is trivially true that) $z = 0 = |z|(\cos(0) + i\sin(0))$.

- (Case 2). Suppose $z \neq 0$ and $\mathsf{Re}(z) \geq 0$.

  Note that $-1 \leq \dfrac{\mathsf{Im}(z)}{|z|} \leq 1$.

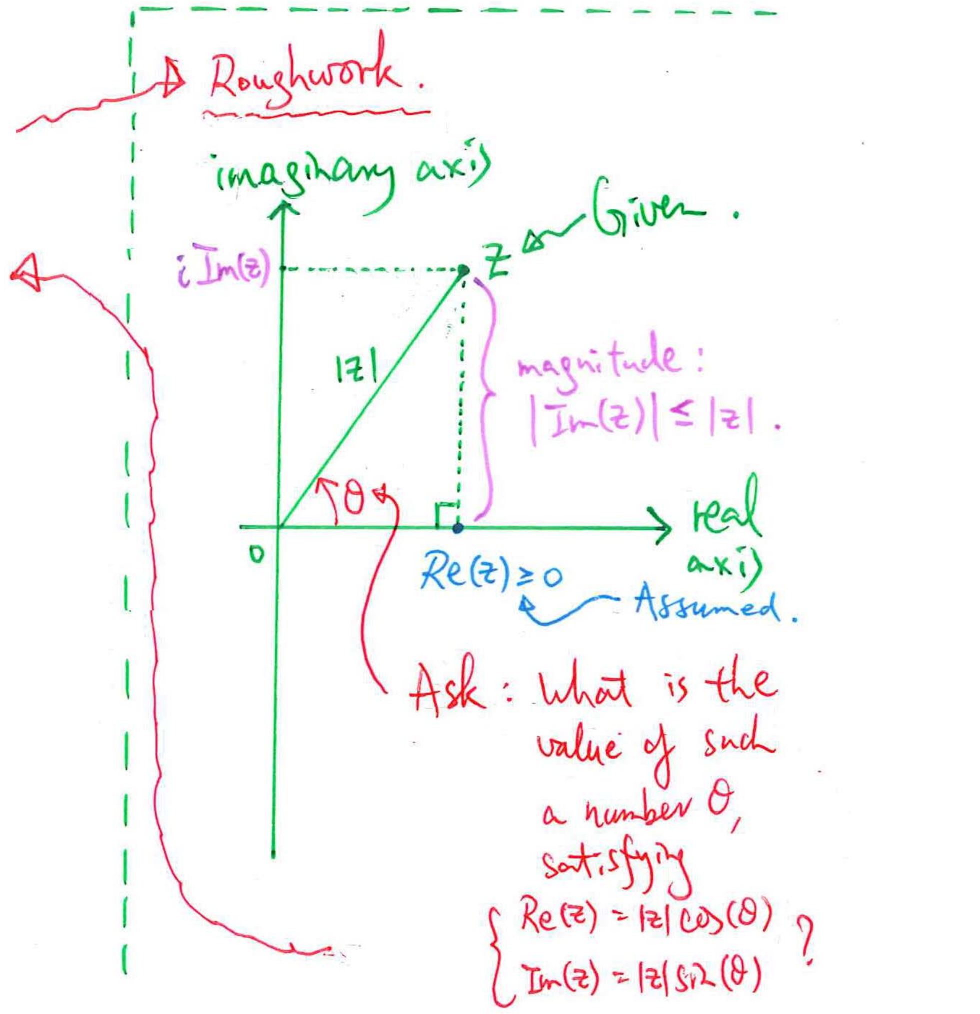
  Define $\theta = \arcsin(\dfrac{\mathsf{Im}(z)}{|z|})$.

  We have $\mathsf{Im}(z) = |z|\sin(\theta)$, and

  $\mathsf{Re}(z) = |z|\cos(\theta)$.

  Then $z = |z|(\cos(\theta) + i\sin(\theta))$.

- $\cdots$



Proof of Theorem (1).

Suppose z be a complex number.

Then we have $z = 0$ or ($z \neq 0$ and $\mathsf{Re}(z) \geq 0$) or ($z \neq 0$ and $\mathsf{Re}(z) < 0$).

- (Case 1). $\cdots$
- (Case 2). $\cdots$
- (Case 3). Suppose $z \neq 0$ and $\mathsf{Re}(z) < 0$.

  Define $w = -z$.

  We have $w \neq 0$ and $\mathsf{Re}(w) > 0$.

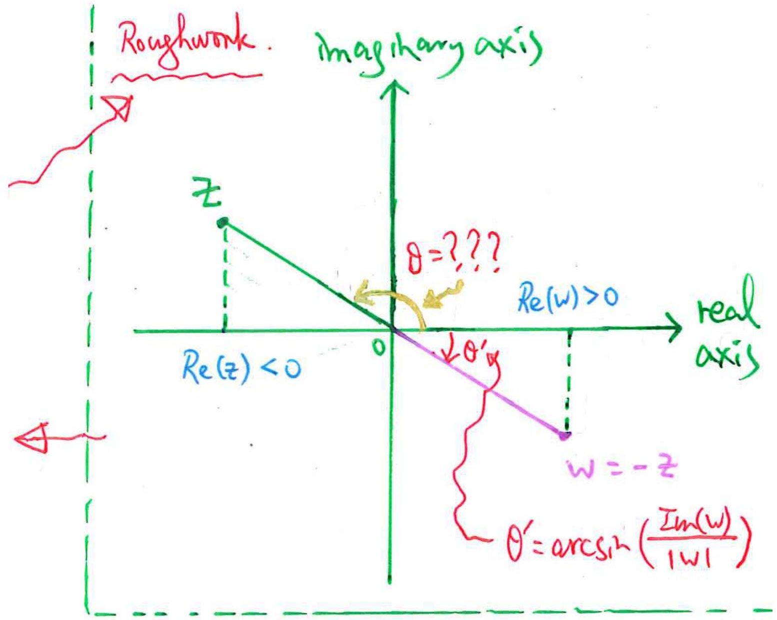
  By the argument for (Case 2),

  there exists some $\theta' \in \mathbb{R}$, namely $\theta' = \arcsin(\frac{\mathsf{Im}(w)}{|w|})$,

  such that $w = |w|(\cos(\theta') + i\sin(\theta'))$.

  Define $\theta = \theta' + \pi$. Then

$$z = -w = -|w|(\cos(\theta') + i\sin(\theta'))$$
$$= |w|(-\cos(\theta') - i\sin(\theta')) = |z|(\cos(\theta) + i\sin(\theta)).$$



3. **Definition. (Arguments and principal argument for a complex number.)**

Let z be a non-zero complex number, and θ be a real number.

Suppose the equality $z = |z|(\cos(\theta) + i\sin(\theta))$ holds.

Then θ is said to be an **argument** *for z.*

Further suppose $-\pi < \theta \leq \pi$.

Then θ is called the **principal argument** *for z, and we write $\theta = \arg(z)$.*

Remark.

- We do not write

  '*the* argument of the (non-zero) complex number so-and-so',

  because the same non-zero complex number has 'infinitely many' different arguments. (For example, for each $n \in \mathbb{Z}$, $2n\pi$ is an argument of 1.)

- However, each non-zero complex number has exactly one principal argument.

  So we should write

  '*the* principal argument of the (non-zero) complex number so-and-so'.

4. **Theorem (2). (Multiplication and division for complex numbers in polar form.)**

Suppose z, w are non-zero complex numbers, with arguments θ, φ respectively. Then:

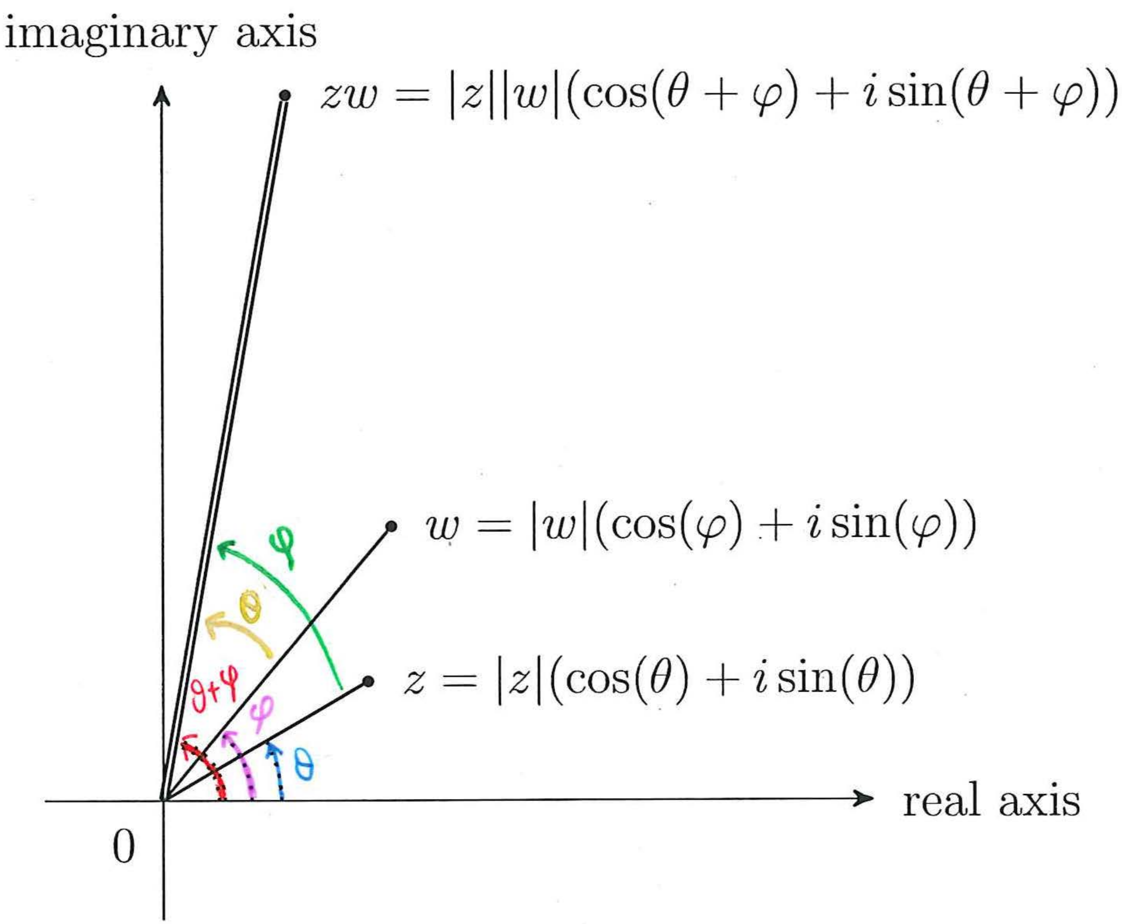
(a) $zw = |z||w|(\cos(\theta + \varphi) + i \sin(\theta + \varphi))$, *and*
$\dfrac{z}{w} = \dfrac{|z|}{|w|}(\cos(\theta - \varphi) + i \sin(\theta - \varphi))$.

(b) *The modulus of zw is $|z||w|$, and the modulus of $\dfrac{z}{w}$ is $\dfrac{|z|}{|w|}$.*

(c) *$\theta + \varphi$ is an argument for zw, and $\theta - \varphi$ is an argument for $\dfrac{z}{w}$.*

Proof of Theorem (2). Exercise. (Apply the 'compound-angle formulae' for the sine and cosine functions.)

Pictorial visualization of multiplication for complex numbers.



5. **Corollary to Theorem (2).**

Suppose z is a non-zero complex number, and θ is an argument of z.

Then $z^2 = |z|^2(\cos(2\theta) + i\sin(2\theta))$.

6. **Definition. (Square roots of a complex number.)**

Let w, z be complex numbers.

We say that z is a square root of w if $w = z^2$.

Examples.

(a) $i, -i$ are square roots of -1.

(b) $1, -1$ are square roots of 1.

(c) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ are square roots of i.

(d) $\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, are square roots of $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Warning.

As these examples illustrate, there may be more than one square root for a given complex number.

In fact, as Theorem (3) will tell us, whenever w is a non-zero complex number, w will have exactly two square roots.

It is not apparent whether any should be privileged over any other.

For this reason:—

- Never write

  $$\text{‘}\textit{the}\text{ square root of the complex number } w\text{’}$$

  unless you are referring to a specific square root of the complex number w that you have already pinpointed.

- Never write

  $$\text{‘}\sqrt{w}\text{’}$$

  unless w is a non-negative real number.
  (When w is a non-negative real number, we ‘privilege’ its non-negative square root over its negative square root.)

7. **Theorem (3).**

Let w, ζ be non-zero complex numbers, and φ be an argument of w.

Suppose $w = \zeta^2$.

Then $\zeta = \sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$ or $\zeta = -\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$.

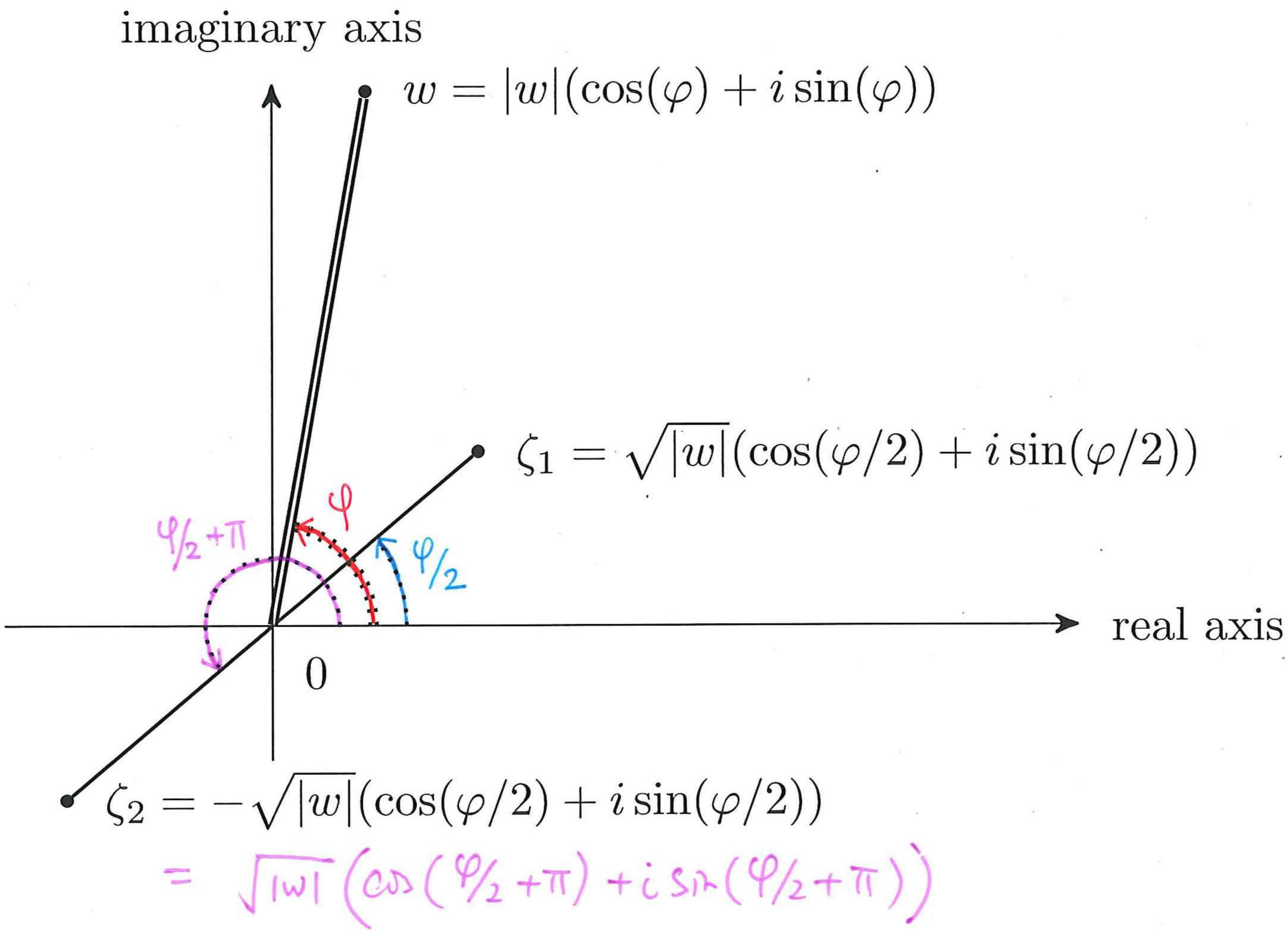
Remark.

Hence the square roots of w are the two numbers

$$\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right), \qquad -\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right).$$

Proof of Theorem (3). Apply Corollary to Theorem (2).

8. **Geometric interpretation of extraction of square roots.**



9. **Theorem (4). (Roots of quadratic polynomials with complex coefficients.)**

Let a, b, c be complex numbers, with $a \neq 0$. Let α be a number. Let $f(z)$ be the quadratic polynomial given by $f(z) = az^2 + bz + c$.

(a) *Suppose α is a root of $f(z)$. Let $\beta = -\dfrac{b}{a} - \alpha$. Then the statements below hold:*

i. *$f(z) = a(z - \alpha)(z - \beta)$ as polynomials.*

ii. *β is a root of $f(z)$.*

iii. *$\alpha\beta = \dfrac{c}{a}$.*

(b) *Define $\Delta_f = b^2 - 4ac$. We call Δ_f the discriminant of the polynomial $f(z)$. Then the statements below hold:*

i. *$f(z) = a\left[\left(z + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right]$ as polynomials.*

(This polynomial equality is referred as 'completing the square for the quadratic polynomial $f(z)$'.)

ii. *Suppose $\Delta_f \neq 0$. Suppose σ is a square root of $\dfrac{\Delta_f}{4a^2}$. Define $\alpha_\pm = -\dfrac{b}{2a} \pm \sigma$ respectively. Then $f(z)$ has two distinct roots amongst the complex numbers, namely α_+, α_-, and $f(z)$ is completely factorized as $f(z) = a(z - \alpha_+)(z - \alpha_-)$.*

iii. *Now suppose $\Delta_f = 0$ instead. Then $f(z)$ has a repeated root, namely, $-b/2a$, amongst the complex numbers, and $f(z)$ is completely factorized as $f(z) = a(z + b/2a)^2$.*

Remark.

What the above result says is that each quadratic polynomial with complex coefficients $f(z)$ has a pair of roots and 'factorizes into a pair of linear polynomials'. Moreover, if the polynomial $f(z)$ is given by $f(z) = az^2 + bz + c$ and the pair of roots concerned are α, β, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Furthermore, regarding the quadratic equation

$$az^2 + bz + c = 0 \quad \text{———} \quad (\star)$$

with unknown x, there are exactly two mutually exclusive possibilities:

(1) Suppose $\Delta_f \neq 0$. Then the equation $(\star)$ has exactly two distinct solutions amongst the complex numbers.

(2) Suppose $\Delta_f = 0$. Then the equation $(\star)$ has exactly one repeated solution amongst the complex numbers.

In any case, the equation $(\star)$ has at least one solution amongst the complex numbers.

Proof of Theorem (4). Exercise. (Generalize what you have learnt about quadratic polynomials with real coefficients in school maths.)

10. **Appendix: Complex numbers and polynomials.**

Theorem (4) is significant in two ways:

(a) This result for quadratic polynomials is a 'baby case' of the **Fundamental Theorem of Algebra**, first proved by Gauss, which says:—

Every non-constant polynomial with coefficients in complex numbers has at least one root amongst the complex numbers.

In fact Gauss gave several proofs for this result.

(You will probably learn one proof in your *complex variables* course.)

(b) We can express all the roots of every quadratic polynomial with coefficients in complex numbers in terms of its coefficients with the help of the operations $+, -, \times, \div$, and with the taking of (square) roots.

So it is natural to ask whether we can do the same thing for cubic polynomials, quartic polynomials, quintic polynomials et cetera.

The answer is *yes* for cubic polynomials and quartic polynomials, but *no* in general for higher-degree polynomials.

(You will know more about these in your *abstract algebra* course.)