1. Since school mathematics we have tacitly accepted that it makes sense to talk about complex numbers:

   A complex number is a mathematical object of the form $a + bi$, in which a, b are some real numbers, and i is a 'special number' (independent of a, b) satisfying the relation $i^2 = -1$.

   We have accepted that it makes sense to talk about sums, differences, products, and quotients for such numbers:

   Let a, b, c, d be real numbers. Consider the complex numbers $a + bi$, $c + di$.

   (a) *We define the number $(a+bi)+(c+di)$, which is called the sum of $a+bi$ and $c+di$, as $(a + c) + (b + d)i$.*

   (b) *We define the number $(a+bi)-(c+di)$, which is called the difference of $a+bi$ from $c+di$, as $(a - c) + (b - d)i$.*

   (c) *We define the number $(a + bi) \cdot (c + di)$, which is called the product of $a + bi$ and $c + di$, as $(ac - bd) + (ad + bc)i$.*

   (d) *Provided that $c \neq 0$ or $d \neq 0$, we define the number $(a + bi) \div (c + di)$, which is called the quotient of $a + bi$ by $c + di$ as $\dfrac{ac + bd}{c^2 + d^2} + \dfrac{-ad + bc}{c^2 + d^2}i$.*

For the moment we stick to this point of view from school maths.

We collect all complex numbers together to form the 'set of all complex numbers'. This set is denoted by $\mathbb{C}$.

The set of all real numbers $\mathbb{R}$ is regarded as a subset of $\mathbb{C}$, through the identification of each real number a as the complex number $a + 0i$.

For each real number b, the number $0 + bi$ is called a purely imaginary number, and can be written simply as bi.

Suppose c, d are real numbers. Then:—

- $0 - (c + di)$ is denoted by $-(c + di)$.
- When it is defined, $1 \div (c + di)$ can be denoted by $(c + di)^{-1}$, or $\dfrac{1}{c + id}$.

2. ‘Laws of arithmetic’ for complex numbers.

Assuming the validity of ‘usual laws of arithmetic’ for real numbers, we can deduce the analogous ‘usual laws of arithmetic’ for complex numbers from the definition of addition, subtraction, multiplication, division for complex numbers we can deduce these ‘algebraic laws’ below.

Theorem (1).

(a) *For any* $z, w \in \mathbb{C}$, $z + w \in \mathbb{C}$.

(b) *For any* $z, w, u \in \mathbb{C}$, $(z + w) + u = z + (w + u)$.

(c) *There exists some* $\nu \in \mathbb{C}$, *namely* $\nu = 0$, *such that for any* $z \in \mathbb{C}$, $\nu + z = z$ *and* $z + \nu = z$.

(d) *For any* $z \in \mathbb{C}$, *there exists some* $w \in \mathbb{C}$, *namely* $w = -z$, *such that* $z+w = 0 = w+z$.

(e) *For any* $z, w \in \mathbb{C}$, $z + w = w + z$.

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(e) *For any* $z, w \in \mathbb{C}$, $z + w = w + z$.

(f) *For any* $z, w \in \mathbb{C}$, $zw \in \mathbb{C}$.

(g) *For any* $z, w, u \in \mathbb{C}$, $(zw)u = z(wu)$.

(h) *There exists some* $\lambda \in \mathbb{C}$, *namely* $\lambda = 1$, *such that for any* $z \in \mathbb{C}$, $\lambda z = z$ *and* $z\lambda = z$.

(i) *For any* $z \in \mathbb{C}$, *if* $z \neq 0$, *then there exists some* $w \in \mathbb{C}$, *namely* $w = \dfrac{1}{z}$, *such that* $zw = 1$ *and* $wz = 1$.

(j) *For any* $z, w \in \mathbb{C}$, $zw = wz$.

(k) *For any* $z, w, u \in \mathbb{C}$, $z(w + u) = zw + zu$.

(l) *For any* $z, w, u \in \mathbb{C}$, $(w + u)z = wz + uz$.

With the help of Theorem (1), we can deduce all other usual 'algebraic laws' for complex numbers which are analogous to those for real numbers.

3. **Digression on polynomials, vectors and matrices.**

We can then further generalize everything that you have learnt about polynomials in school maths, up to the point where *calculus* is involved.

You can also generalize almost everything that you have learnt in your *linear algebra* course about vectors and matrices with real entries to vectors and matrices with complex entries, as long as the material does not involve inequalities for real numbers.

(What cannot be immediately adapted from real to complex is mostly from the topics *eigenvalue and eigenvectors* and *inner product*.)

This is done by replacing the word *real* by the word *complex* and replacing the symbol $\mathbb{R}$ by the symbol $\mathbb{C}$ in every definition/result concerned with polynomials, vectors and matrices.

4. **Definition. (Real part, imaginary part, conjugate, modulus.)**
Let z be a complex number.
Denote the real part and the imaginary part of z by $\mathsf{Re}(z)$, $\mathsf{Im}(z)$ respectively.
(So $z = \mathsf{Re}(z) + i\mathsf{Im}(z)$.)

(a) *The* **complex conjugate** *of z is defined to be the complex number*

$$\mathsf{Re}(z) - i\mathsf{Im}(z).$$

It is denoted by $\overline{z}$.

(b) *The* **modulus** *of z is defined to be the non-negative real number*

$$\sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}.$$

It is denoted by $|z|$.

All results stated below are immediate consequences of the definition for the notions of *real part*, *imaginary part*, *conjugate* and *modulus*. Their proofs are left as exercises.

Many of these results can be interpreted in geometric terms. (For detail, refer to the handout *Argand plane*.)

5. **Theorem (2).**

Let z, w be complex numbers. The statements below hold:

(a) $\mathsf{Re}(z+w) = \mathsf{Re}(z) + \mathsf{Re}(w)$ *and* $\mathsf{Im}(z+w) = \mathsf{Im}(z) + \mathsf{Im}(w)$.

(b) $\mathsf{Re}(zw) = \mathsf{Re}(z)\mathsf{Re}(w) - \mathsf{Im}(z)\mathsf{Im}(w)$ *and* $\mathsf{Im}(zw) = \mathsf{Re}(z)\mathsf{Im}(w) + \mathsf{Im}(z)\mathsf{Re}(w)$.

6. **Theorem (3).**

Let z, w be complex numbers. The statements below hold:

(a) $\mathsf{Re}(z) = \dfrac{z+\overline{z}}{2}$ *and* $\mathsf{Im}(z) = \dfrac{z-\overline{z}}{2i}$.

(b) $\mathsf{Re}(\overline{z}) = \mathsf{Re}(z)$ *and* $\mathsf{Im}(\overline{z}) = -\mathsf{Im}(z)$.

(c) $\overline{\overline{z}} = z$.

(d) $\overline{z+w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z} \cdot \overline{w}$.

7. **Theorem (4).**

Let z, w be complex numbers. The statements below hold:

(a) $|\overline{z}| = |z|$.

(b) $|z|^2 = z \cdot \overline{z}$.

(c) $|zw| = |z| \cdot |w|$.

(d) $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$.

Remark. The equality described in item (d) is called the 'parallelogramic identity'.

8. **Theorem (5).**

The statements below hold:

(a) *Suppose z is a complex number.*
Then $|\mathsf{Re}(z)| \leq |z|$.
Equality holds iff z is a real number.

(b) *Suppose z is a complex number.*
Then $|\mathsf{Im}(z)| \leq |z|$.
Equality holds iff z is a purely imaginary number.

9. **Definition. (Real multiples.)**

Let z, w be complex numbers.

We say that w is a real multiple of z if there exists some $t \in \mathbb{R}$ such that $w = tz$.

Furthermore:—

- *When t is non-zero, we say that w is a non-zero real multiple of z.*
- *When t is non-negative, we say that w is a non-negative (real) multiple of z.*
- *When t is positive, we say that w is a positive (real) multiple of z.*

10. **Lemma (6).**

Let z, w be complex numbers. The statements below hold:

(a) *$z\overline{w}$ is a real number iff one of z, w is a real multiple of the other.*

(b) *$z\overline{w}$ is a non-zero real number iff one of z, w is a non-zero real multiple of the other.*

(c) *$z\overline{w}$ is a non-negative real number iff one of z, w is a non-negative multiple of the other.*

(d) *$z\overline{w}$ is a positive real number iff one of z, w is a positive multiple of the other.*

11. **Lemma (7).**

Suppose z, w are complex numbers.

Then $|\mathsf{Re}(z\overline{w})| \leq |z||w|$.

Equality holds iff at least one of z, w is a real multiple of the other.

12. With the help of Lemma (6) and Lemma (7), we can deduce Theorem (8), Theorem (9), Theorem (10).

They generalize the Triangle Inequality on the 'real line' and its consequences, which are introduced in the handout *Absolute Value and Triangle Inequality for the Reals*.

Theorem (8). (Triangle Inequality on the 'complex plane'.)

Suppose z, w are complex numbers.

Then $|z + w| \leq |z| + |w|$.

Equality holds iff at least one of z, w is a non-negative multiple of the other.

Theorem (9). (Triangle Inequality on the 'complex plane', also.)
Suppose z, w are complex numbers.

Then $\Big|\, |z| - |w| \,\Big| \leq |z - w|$.

Equality holds iff at least one of z, w is a non-negative multiple of the other.

Theorem (10). (Generalization of Theorem (8) to the 'many number' situation.)
Let n be an integer no less than 2.

Suppose $z_1, z_2, \cdots, z_n$ are complex numbers.

Then $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

Equality holds iff there is some k amongst $1, 2, \cdots, n$ such that each of $z_1, z_2, \cdots, z_n$ is a non-negative multiple of z_k.