MATH1050 Examples: Mathematical induction.

1. Apply mathematical induction to justify each of the statements below:

   (a) $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1}$ *for any* $n\in\mathbb{N}\backslash\{0\}$.

   (b) $-1^2+2^2-3^2+4^2+\cdots+(-1)^n n^2=(-1)^n\cdot\dfrac{n(n+1)}{2}$ *whenever n is a positive integer.*

   (c) $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\cdot\ldots\cdot\left(1-\dfrac{1}{n^2}\right)=\dfrac{n+1}{2n}$ *for any* $n\in\mathbb{N}\backslash\{0,1\}$.

   (d) $\dfrac{0}{2^0}+\dfrac{1}{2^1}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}=2-\dfrac{n+2}{2^n}$ *for any* $n\in\mathbb{N}$.

   (e) $(1^2+1)\cdot(1!)+(2^2+2)\cdot(2!)+(3^2+3)\cdot(3!)+\cdots(n^2+1)\cdot(n!)=n\cdot[(n+1)!]$ *for any* $n\in\mathbb{N}\backslash\{0\}$.

   (f) $0^2+1^2+2^2+\cdots+(2n-1)^2=\dfrac{n(2n-1)(4n-1)}{3}$ *for each positive integer n.*

   (g) $\displaystyle\sum_{k=0}^{n}k^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$ *for any* $n\in\mathbb{N}$.

   (h) $\displaystyle\sum_{k=0}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ *for any* $n\in\mathbb{N}$.

   (i) $\displaystyle\sum_{k=1}^{2n}\frac{1}{k(k+1)}=\frac{2n}{2n+1}$ *for any positive integer n.*

   (j) $\displaystyle\sum_{k=2}^{n}\binom{n}{2}=\binom{n+1}{3}$ *for any integer n greater than 1.*

2. Apply mathematical induction to prove the statement below.

   - *Suppose α is a number, not equal to 1.*
     Then $\displaystyle\sum_{k=1}^{n}k\alpha^{k-1}=\frac{1-(n+1)\alpha^n+n\alpha^{n+1}}{(1-\alpha)^2}$ *for each positive integer n.*

3. (a) Suppose θ,α are real numbers.

   i. Verify that $\cos(\theta+k\alpha)\sin(\alpha)=\dfrac{1}{2}\left(\sin(\theta+\dfrac{2k+1}{2}\alpha)-\sin(\theta+\dfrac{2k-1}{2}\alpha)\right)$ for each integer k.

   ii.$^\diamond$ Now suppose $\sin(\dfrac{\alpha}{2})\neq 0$ also. By applying mathematical induction, or otherwise, prove that
   $$\sum_{k=0}^{n}\cos(\theta+k\alpha)=\frac{\cos(\theta+n\alpha/2)\sin((n+1)\alpha/2)}{\sin(\alpha/2)}\text{ for each } n\in\mathbb{N}.$$

   (b) Suppose $\sin(\beta)\neq 0$. By applying the results above, or otherwise, prove that
   $\displaystyle\sum_{k=1}^{2m}\cos^2(k\beta)=\frac{\cos(Am\beta)\sin((Bm+C)\beta)}{D\sin(\beta)}+\frac{Em+F}{2}$ and $\displaystyle\sum_{k=1}^{2m}\sin^2(k\beta)=\frac{\cos(Am\beta)\sin((Bm+C)\beta)}{D\sin(\beta)}+\frac{Gm+H}{2}$ for each positive integer m.
   Here A,B,C,D,E,F,G,H are integers whose respective values you have to determine explicitly.

4. Apply mathematical induction to prove the statement below:

   - *Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by*
   $$\begin{cases} a_1 &= 0\\ a_{n+1} &= 2n-a_n \quad \text{if} \quad n\geq 1\end{cases}.$$
     Then $a_n=n+\dfrac{(-1)^n-1}{2}$ *for each positive integer n.*

5. Apply mathematical induction to prove the statement below:

   - *Let a, b be distinct positive real numbers, and $\{c_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by*

$$\left\{ \begin{array}{lcl} c_1 & = & a+b \\ c_{n+1} & = & a+b-\dfrac{ab}{c_n} \quad \text{if} \quad n \geq 1 \end{array} \right. .$$

   Then $c_n = \dfrac{a^{n+1}-b^{n+1}}{a^n-b^n}$ for each positive integer n.

6. Prove the statement below:

   - *Let α, β are the two distinct roots of the polynomial $f(x) = x^2 - 2x - 1$. Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by*

$$\left\{ \begin{array}{llllll} a_1 & = & 1, & a_2 & = & 3, \\ & & & a_{n+2} & = & 2a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{array} \right. .$$

   Then $a_n = \dfrac{1}{2}(\alpha^n + \beta^n)$ for each positive integer n.

   Remark. You have to think carefully which proposition is to be formulated and proved by mathematical induction.

7. Apply mathematical induction to prove the statement below:

   - *Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence of real numbers defined by*

$$\left\{ \begin{array}{lllllllll} a_0 & = & 1, & a_1 & = & 6, & a_2 & = & 45, \\ & & & & & & a_{n+3} & = & 9a_{n+2} - 27a_{n+1} + 27a_n \quad \text{if} \quad n \geq 0 \end{array} \right. .$$

   Then $a_n = 3^n(n^2+1)$ for each $n \in \mathbb{N}$.

   Remark. You have to think carefully which proposition is to be formulated and proved by mathematical induction.

8. Apply mathematical induction to prove the statement below:

   - *Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence in $\mathbb{N}$. Suppose $n \leq \sum_{j=1}^{n} {a_j}^2 \leq n + 1 + (-1)^n$ for each positive integer n. Then $a_n = 1$ for each positive integer n.*

   Remark. You have to think carefully which proposition is to be formulated and proved by mathematical induction.

Answer.

4. (a) ——

   (b) $\sum_{k=1}^{2m} \cos^2(k\beta) = \dfrac{\cos(2m\beta)\sin((2m+1)\beta)}{2\sin(\beta)} + \dfrac{2m-1}{2}$ and $\sum_{k=1}^{2m} \sin^2(k\beta) = \dfrac{\cos(2m\beta)\sin((2m+1)\beta)}{2\sin(\beta)} + \dfrac{2m+1}{2}$