# MATH1050 Principle of Mathematical Induction and Well-ordering Principle for Integers.

1. Recall the **the Principle of Mathematical Induction (UPMI)**:

   *Let $P(n)$ be a predicate with variable $n$.*

   *Suppose the statement $P(0)$ is true.*

   *Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.*

   *Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.*

   Also recall the definition for the notion of *least element*:

   *Suppose $A$ is a subset of $\mathbb{R}$, and $\lambda \in A$. Then $\lambda$ is said to be a least element of $A$ if for any $x \in A$, $\lambda \leq x$.*

   Further recall the **Well-ordering Principle for Integers (WOPI)**:

   *Let $S$ be a subset of $\mathbb{N}$. Suppose $S$ is non-empty. Then $S$ has a least element.*

   We are going to explain why (WOPI) and (UPMI) are logically equivalent.

   As preparation, we have to introduce two statements respectively known as the Set-theoretic Formulation of the Principle of Mathematical Induction and the Set-theoretic Formulation of the Second Principle of Mathematical Induction.

2. **Principle of Mathematical Induction, (set-theoretic formulation). (SPMI.)**

   *Let $T$ be a subset of $\mathbb{N}$.*

   *Suppose $0 \in T$.*

   *Further suppose that for any $k \in \mathbb{N}$, if $k \in T$ then $k+1 \in T$.*

   *Then $T = \mathbb{N}$.*

   **Second Principle of Mathematical Induction, (set-theoretic formulation). (SPMI2.)**

   *Let $U$ be a subset of $\mathbb{N}$.*

   *Suppose $0 \in U$.*

   *Further suppose that for any $k \in \mathbb{N}$, if $0, 1, 2, \cdots, k \in U$ then $k+1 \in U$.*

   *Then $U = \mathbb{N}$.*

3. **Logical equivalence between the Principle of Mathematical Induction and the Well-ordering Principle.**

   The logical equivalence of the statements (UPMI) and (WOPI) is a consequence of Theorem (1) and Theorem (2).

   **Theorem (1).**

   *The statements (UPMI), (SPMI) are logically equivalent.*

   **Theorem (2).**

   *The statements (WOPI), (SPMI), (SPMI2) are logically equivalent.*

4. **Proof of Theorem (1).**

   Argument for '(SPMI)$\Longrightarrow$(UPMI)':

   Assume (SPMI) holds:

   * *Let $S$ be a subset of $\mathbb{N}$. Suppose $0 \in S$. Further suppose that for any $k \in \mathbb{N}$, if $k \in S$ then $k+1 \in S$. Then $S = \mathbb{N}$.*

   [We want to deduce from this assumption '(SPMI) holds' that (UPMI) holds.]

   Let $P(n)$ be a predicate with variable $n$. Suppose the statement $P(0)$ is true. Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true. [We want to deduce, by applying (SPMI), that for any $n \in \mathbb{N}$, the statement $P(n)$ is true.]

   Define $S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. [We now proceed to prove that $S = \mathbb{N}$.]

   - Since the statement $P(0)$ is true, we have $0 \in S$.
     Pick any $k \in \mathbb{N}$. Suppose $k \in S$. Then (by the definition of $S$) the statement $P(k)$ is true. Since $P(k)$ is true, $P(k+1)$ is also true. Therefore (by the definition of $S$) we have $k+1 \in S$.
     Now, by (SPMI), $S = \mathbb{N}$.

It follows that for any $n \in \mathbb{N}$, the statement $P(n)$ is true.

Argument for '(UPMI)$\Longrightarrow$(SPMI)':

Assume (UPMI) holds:

* *Let $P(n)$ be a predicate with variable $n$. Suppose the statement $P(0)$ is true. Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true. Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.*

[We want to deduce from this assumption '(UPMI) holds' that (SPMI) holds.]

Let $S$ be a subset of $\mathbb{N}$. Suppose $0 \in S$. Further suppose that for any $k \in \mathbb{N}$, if $k \in S$ then $k+1 \in S$. [We want to deduce, by applying (UPMI), that $\mathbb{N} \subset S$.]

- For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition $n \in S$. [We now proceed to apply mathematical induction to prove that for any $n \in \mathbb{N}$, $P(n)$ is true.]
  By assumption, $0 \in S$. Then $P(0)$ is true.
  Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $k \in S$. By the assumption on $S$, since $k \in S$, we also have $k+1 \in S$. Therefore $P(k+1)$ is true.
  By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

We have verified that every element of $\mathbb{N}$ is an element of $S$. Then $\mathbb{N} \subset S$.
By definition $S \subset \mathbb{N}$. Hence $S = \mathbb{N}$.

The result follows.

5. **Outline of Proof of Theorem (2).** Here we give the set-up and the beginning steps for each part of the argument. Fill in the remaining steps as an exercise.

Argument for '(WOPI)$\Longrightarrow$(SPMI)':

- Suppose the statement (WOPI) holds.
  Let $T$ be a subset of $\mathbb{N}$. Suppose $0 \in T$. Further suppose that for any $n \in \mathbb{N}$, if $k \in T$ then $k+1 \in T$.
  By definition, $T \subset \mathbb{N}$. We verify that $\mathbb{N} \subset T$, with the method of proof by contradiction.

  [*Idea.* Suppose it were true that $\mathbb{N} \not\subset T$. We look for a contradiction.
  By assumption, there would exist some $x_0 \in \mathbb{N}$ such that $x_0 \notin T$.
  Define $S = \{x \in \mathbb{N} : x \notin T\}$. Since $x_0 \in S$, we would have $S \neq \emptyset$.
  (WOPI) would tell us that $S$ had a least element, say, $\lambda$. By studying the number $\lambda$, we will come to a desired contradiction.
  We will be forced to conclude that $\mathbb{N} \subset T$ in the first place.]

  Since $T \subset \mathbb{N}$ and $\mathbb{N} \subset T$, we have $T = \mathbb{N}$.
  It follows that the statement (SPMI) holds.

Argument for '(SPMI)$\Longrightarrow$(SPMI2)':

- Suppose that the statement (SPMI) holds.
  Let $U$ be a subset of $\mathbb{N}$. Suppose that $0 \in U$. Further suppose that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$.
  By definition, $U \subset \mathbb{N}$. We verify that $\mathbb{N} \subset U$.

  [*Idea.* Define $T = \{x \in \mathbb{N} : [\![0, x]\!] \subset U\}$.
  Apply (SPMI) to deduce that $T = \mathbb{N}$.
  Now ask: Is it true that every element of $T$ an element of $U$? If yes, then it follows that $\mathbb{N} \subset U$, and furthermore $\mathbb{N} = U$.]

  It follows that the statement (SPMI2) holds.

Argument for '(SPMI2)$\Longrightarrow$(WOPI)':

- Suppose that the statement (SPMI2) holds.
  Let $S$ be a non-empty subset of $\mathbb{N}$. We verify that $S$ has a least element with the method of proof by contradiction:

  [*Idea.* Suppose it were true that $S$ did not have a least element. We look for a contradiction.
  Define $U = \mathbb{N} \backslash S$.
  We verify that $0 \in U$, and that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$.
  (SPMI2) would then imply that $U = \mathbb{N}$. It would then follow that $S = \emptyset$: this is the desired contradiction.]

It follows that the statement (WOPI) holds.

The result follows.

6. Just as there are one formulation of the Principle of Mathematical Induction with set-theoretic jargon and one without, there are also one formulation of the Second Principle of Mathematical Induction with set-theoretic jargon and one without.

   **Second Principle of Mathematical Induction. (UPMI2.)**

   *Let $Q(n)$ be a predicate with variable $n$.*

   *Suppose the statement $Q(0)$ is true.*

   *Further suppose that for any $k \in \mathbb{N}$, if the statements $Q(0), Q(1), \cdots, Q(k)$ are true then the statement $Q(k+1)$ is true.*

   *Then the statement $Q(n)$ is true for any $n \in \mathbb{N}$.*

   **Theorem (3).**

   *The statements (SPMI2), (UPMI2) are logically equivalent.*

   **Proof of Theorem (3).** Exercise.

7. The Second Principle of Mathematical Induction makes possible the method of argument, usually known as 'strong induction', for statements in the form

   '*For any natural number $n$, $Q(n)$ is true*'

   in which '$Q(x)$' is a predicate with the variable $x$. The general scheme is described below:

   - **Step (0).** Identify $Q(n)$ and write it down explicitly.
   - **Step (1).** Prove the statement $Q(0)$. (This is the 'initial step argument'.)
   - **Step (2).** Assume the statements $Q(0), Q(1), Q(2), \cdots, Q(k)$ to be true. (This is called the induction assumption.) Prove the statement $Q(k+1)$ under this assumption. (This is the 'induction argument'.)
   - **Step (3).** Declare that according to the Second Principle of Mathematical Induction, $Q(n)$ is true for any $n \in \mathbb{N}$.