MATH1050 Examples on argument by mathematical induction

1. Recall the notion of *predicate*:

   A **predicate with variables** $x, y, z, \cdots$ *is a statement 'modulo' the ambiguity of possibly one or several variables* $x, y, z, \cdots$.
   Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

   Also recall the statement (UPMI) (together with the logically equivalent statement (VPMI)) from the handout *Argument by mathematical induction*:

   (First) Principle of Mathematical Induction (in its 'usual' formulation). (UPMI).
   Let $P(n)$ be a predicate with variable n.
   Suppose the statement $P(0)$ is true.
   Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.
   Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.

   Principle of Mathematical Induction, (variant of its 'usual' formulation). (VPMI).
   Let $R(n)$ be a predicate with variable n. Let M be an integer.
   Suppose the statement $R(M)$ is true.
   Further suppose that for any $k \in [\![M, +\infty)$, if the statement $R(k)$ is true then the statement $R(k+1)$ is true.
   Then the statement $R(n)$ is true for any $n \in [\![M, +\infty)$.

   Here we give some examples on argument by mathematical induction. As explained in the handout *Argument by mathematical induction*, each of them has to follow a certain format, as dictated by the role played by the statement (UPMI) (or the statement (VPMI)) in such an argument.

2. **Example (A).**
   We want to verify the statement

   $(\star)$ $0^3 + 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ *for any $n \in \mathbb{N}$.*

   Observation. Note that $(\star)$ is of the form

   '*for any $n \in \mathbb{N}$, $P(n)$ is true*',

   in which $P(n)$ is the predicate with variable n that reads:

   '$0^3 + 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$'

   So it makes sense to attempt to argue for $(\star)$ by mathematical induction.

   Justification of $(\star)$ (by mathematical induction).

   - For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition
   $$0^3 + 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$
   - $0^3 = 0 = \dfrac{0^2(0+1)^2}{4}$. Then $P(0)$ is true.
   - Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $0^3 + 1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.

     We prove that $P(k+1)$ is true:

     We have
     $$\begin{aligned} 0^3+1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{(k+1)^2[k^2+4(k+1)]}{4} \\ &= \frac{(k+1)^2[(k+1)+1]^2}{4}. \end{aligned}$$

     Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

3. **Example (B).**

   We want to verify the statement $(\star)$:

   $(\star)$ *$n^3 - n$ is divisible by 3 for any $n \in \mathbb{N}$.*

   Observation. Note that $(\star)$ is of the form

   '*for any $n \in \mathbb{N}$, $P(n)$ is true*',

   in which $P(n)$ is the predicate with variable n that reads:

   '*$n^3 - n$ is divisible by 3*'.

   So it makes sense to attempt to argue for $(\star)$ by mathematical induction.

   Justification of $(\star)$ (by mathematical induction).

   - For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition below:
     $n^3 - n$ is divisible by 3.
   - $0^3 - 0 = 0 = 0 \cdot 3$ and $0 \in \mathbb{Z}$.
     Hence, by definition, $0^3 - 0$ is divisible by 3.
     Then $P(0)$ is true.
   - Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $k^3 - k$ is divisible by 3.
     We prove that $P(k+1)$ is true:

     By definition, there exists some $q \in \mathbb{Z}$ such that $k^3 - k = 3q$.
     We have $(k+1)^3 - (k+1) = (k^3 - k) + 3k^2 + 3k = 3(q + k^2 + k)$.
     Since $q, k \in \mathbb{Z}$, we have $q + k^2 + k \in \mathbb{Z}$.
     Then, by definition, $(k+1)^3 - (k+1)$ is divisible by 3.
     Hence $P(k+1)$ is true.
   - By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

4. **Example (C).**

   We want to verify the statement

   $(\star)$ $\dfrac{(2n)!}{(n!)^2} > \dfrac{4^n}{n+1}$ *for any $n \in \mathbb{N}\backslash\{0, 1\}$.*

   Observation. Note that $(\star)$ is of the form

   '*for any integer n no less than 2, $P(n)$ is true*',

   in which $P(n)$ is the predicate with variable n that reads:

   '$\dfrac{(2n)!}{(n!)^2} > \dfrac{4^n}{n+1}$'.

   So it makes sense to attempt to argue for $(\star)$ by mathematical induction.

   Justification of $(\star)$ (by mathematical induction).

   - For any $n \in \mathbb{N}\backslash\{0, 1\}$, denote by $P(n)$ the proposition
     $$\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}.$$
   - Note that $\dfrac{(2 \cdot 2)!}{(2!)^2} = \dfrac{24}{4} = 6 > \dfrac{16}{3} = \dfrac{4^2}{2+1}$. Then $P(2)$ is true.
   - Let $k \in \mathbb{N}\backslash\{0, 1\}$. Suppose $P(k)$ is true. Then $\dfrac{(2k)!}{(k!)^2} \cdot \dfrac{k+1}{4^k} > 1$.

     We prove that $P(k+1)$ is true:

[We intend to deduce the inequality $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > \dfrac{4^{k+1}}{(k+1)+1}$.]

Note that $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > 0$, $\dfrac{4^{k+1}}{(k+1)+1} > 0$ and

$$\begin{aligned}
\frac{[2(k+1)]!}{[(k+1)!]^2} \cdot \frac{(k+1)+1}{4^{k+1}} &= \frac{(2k+2)(2k+1)}{(k+1)^2} \cdot \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{k+2}{4(k+1)} \\
&= \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{(2k+1)(k+2)}{2(k+1)^2} \\
&> 1 \cdot \frac{(2k+1)(k+2)}{2(k+1)^2} = \frac{2k^2+5k+2}{2k^2+4k+2} > 1
\end{aligned}$$

Then $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > \dfrac{4^{k+1}}{(k+1)+1}$.

Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

5. **Example (D).**

We want to verify the statement

($\star$) *For any $n \in [\![8, +\infty)$, there exist some $u, v \in \mathbb{N}$ such that $n = 3u + 5v$.*

Observation. Note that ($\star$) is of the form

‘*for any integer n no less than 8, $P(n)$ is true*’,

in which $P(n)$ is the predicate with variable n that reads:

‘*there exist some $u, v \in \mathbb{N}$ (dependent on n) such that $n = 3u + 5v$*’.

So it makes sense to attempt to argue for ($\star$) by mathematical induction.

Justification of ($\star$) (by mathematical induction).

- For any $n \in [\![8, +\infty)$, denote by $P(n)$ the proposition below:

  there exist some $u, v \in \mathbb{N}$ such that $n = 3u + 5v$.
- Note that $1 \in \mathbb{N}$ and $8 = 3 \cdot 1 + 5 \cdot 1$. Then $P(8)$ is true.
- Let $k \in [\![8, +\infty)$. Suppose $P(k)$ is true. Then there exist some $u, v \in \mathbb{N}$ such that $k = 3u + 5v$.

  We prove that $P(k+1)$ is true:

  Note that $v = 0$ or $v \geq 1$.
  - (Case 1). Suppose $v = 0$. Then $k = 3u$.
    Since $k \geq 8$, we have $u \geq 3$. Then $u - 3 \in \mathbb{N}$.
    Note that $k + 1 = 3u + 1 = 3(u-3) + 5 \cdot 2$.
  - (Case 2). Suppose $v \geq 1$. Then $v - 1 \in \mathbb{N}$.
    Since $u \in \mathbb{N}$, we have $u + 2 \in \mathbb{N}$.
    Note that $k + 1 = 3u + 5v + 1 = 3(u+2) + 5(v-1)$.

  Hence, in any case $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in [\![8, +\infty)$.

6. **Example (E).**

We want to verify the statement ($\star\star$):

($\star\star$) *Suppose α is a number, not equal to 1. Then $\displaystyle\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}$ for each positive integer n.*

Observation. Note that ($\star\star$) is of the form

‘*Suppose blah-blah-blah. Then for any positive integer n, $P(n)$ is true*’,

in which $P(n)$ is the predicate with variable n that reads:

'$\displaystyle\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}$ '.

It makes sense to attempt to argue for ($\star\star$) by mathematical induction. However, the process of the argument is conducted under the 'overarching assumption'

' *α is a number, not equal to* 1'.

Justification of ($\star\star$) (by mathematical induction).

Suppose α is a number, not equal to 1.

- For any positive integer n, denote by $P(n)$ the proposition

$$\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}.$$

- We have $\displaystyle\sum_{k=1}^{1} \alpha^{k-1} = 1 = \frac{1-\alpha^1}{1-\alpha}$. Hence $P(1)$ is true.
- Let $m \in \mathbb{N}\backslash\{0\}$. Suppose $P(m)$ is true. We deduce that $P(m+1)$ is true:

  We have

$$\sum_{k=1}^{m+1} \alpha^{k-1} = \sum_{k=1}^{m} \alpha^{k-1} + \alpha^m = \frac{1-\alpha^m}{1-\alpha} + \alpha^m = \frac{1-\alpha^m}{1-\alpha} + \frac{\alpha^m - \alpha^{m+1}}{1-\alpha} = \frac{1-\alpha^{m+1}}{1-\alpha}$$

  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

7. **Example (F).**

We want to verify the statement ($\star\star$):

($\star\star$) *Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by*

$$\left\{\begin{array}{lcll} a_1 & = & 0 & \\ a_{n+1} & = & 2n - a_n & \text{if} \quad n \geq 1 \end{array}\right. .$$

Then $a_n = n + \dfrac{(-1)^n - 1}{2}$ for each positive integer n.

Observation. Note that ($\star\star$) is of the form

'*Suppose blah-blah-blah. Then for any positive integer n, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

'$a_n = n + \dfrac{(-1)^n - 1}{2}$'.

It makes sense to attempt to argue for ($\star\star$) by mathematical induction. However, the process of the argument is conducted under the 'overarching assumption'

' *$\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by so-and-so.*'

Justification of ($\star\star$) (by mathematical induction).

Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by

$$\left\{\begin{array}{lcll} a_1 & = & 0 & \\ a_{n+1} & = & 2n - a_n & \text{if} \quad n \geq 1 \end{array}\right. .$$

- For any positive integer n, denote by $P(n)$ the proposition

$$a_n = n + \frac{(-1)^n - 1}{2}.$$

- Note that $a_1 = 0 = 1 + \dfrac{(-1)^1 - 1}{2}$. Then $P(1)$ is true.

- Let k be a positive integer. Suppose $P(k)$ is true. Then

$$a_k = k + \frac{(-1)^k - 1}{2}.$$

We verify that $P(k+1)$ is true:

We have

$$a_{k+1} = 2k - a_k = 2k - \left[k + \frac{(-1)^k - 1}{2}\right] = (k+1) - 1 + \frac{-(-1)^k + 1}{2} = (k+1) + \frac{(-1)^{k+1} - 1}{2}$$

Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

8. **Example (G).**

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Let n be an integer greater than 1. Suppose $u_1, u_2, \cdots, u_n$ are real numbers.*
Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.

Remark. Refer to the handout *Absolute Value and Triangle Inequality for the Reals*. This statement is the 'inequality part' in the Generalization of Triangle Inequality on the real line.

Observation. Note that $(\star\star)$ is of the form

'*for any integer n greater than 1, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

'*Suppose $u_1, u_2, \cdots, u_n$ are real numbers.*
Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.'.

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction.

Justification of $(\star\star)$ (by mathematical induction).

- For any integer n greater than 1, denote by $P(n)$ the proposition below:

  'Suppose $u_1, u_2, \cdots, u_n$ are real numbers. Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.'

- $P(2)$ is an immediate consequence of the Triangle Inequality on the real line.
- Let $k \in \mathbb{N}\backslash\{0, 1\}$. Suppose $P(k)$ holds.

  We verify $P(k+1)$:

  Suppose $u_1, u_2, \cdots, u_k, u_{k+1}$ are real numbers.
  Define $v_1 = u_1$, $v_2 = u_2$, ..., $v_{k-1} = u_{k-1}$, and define $v_k = u_k + u_{k+1}$.
  By $P(k)$ and $P(2)$ in succession, we have

$$\begin{aligned} |u_1 + u_2 + \cdots + u_{k-1} + u_k + u_{k+1}| &= |v_1 + v_2 + \cdots + v_{k-1} + v_k| \\ &\leq |v_1| + |v_2| + \cdots + |v_{k-1}| + |v_k| \\ &= |u_1| + |u_2| + \cdots + |u_{k-1}| + |u_k + u_{k+1}| \\ &\leq |u_1| + |u_2| + \cdots + |u_{k-1}| + |u_k| + |u_{k+1}| \end{aligned}$$

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any integer n greater than 1.

9. **Example (H)**.

We take for granted the validity of the statement known as Euclid's Lemma:

Let $h, k \in \mathbb{Z}$, and p be a prime number. Suppose hk is divisible by p. Then at least one of h, k is divisible by p.

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Suppose p is a prime number. Then, for any integer n greater than 1, for any integers $a_1, a_2, \cdots, a_n$, if $a_1 a_2 \cdot \ldots \cdot a_n$ is divisible by p, then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.*

Remark. We may think of the statement $(\star\star)$ as a generalization of Euclid's Lemma to the product-of-many-integers situation.

Note that $(\star\star)$ is of the form

> '*Suppose blah-blah-blah. Then for any integer n greater than 1, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

> '*for any integers $a_1, a_2, \cdots, a_n$, if $a_1 a_2 \cdot ... \cdot a_n$ is divisible by p, then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p*'.

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction. However , the process of the argument is conducted under the 'overarching assumption'

> ' *p is a prime number.*'

Justification of $(\star\star)$ (by mathematical induction).

- Let p be a prime number. For any $n \in \mathbb{N}\backslash\{0,1\}$, denote by $P(n)$ the following proposition:

  Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_n$ is divisible by p. Then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.
- $P(2)$ is an immediate consequence of Euclid's Lemma.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. (Therefore, for any $c_1, c_2, \cdots, c_k \in \mathbb{Z}$, if $c_1 c_2 \cdot ... \cdot c_k$ is divisible by p then at least one of $c_1, c_2, \cdots, c_k$ is divisible by p.)

  We prove that $P(k+1)$ is true:

  Let $a_1, a_2, \cdots, a_k, a_{k+1} \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_k a_{k+1}$ is divisible by p.
  Note that $a_1 a_2 \cdot ... \cdot a_k \in \mathbb{Z}$ and $(a_1 a_2 \cdot ... \cdot a_k) a_{k+1} = a_1 a_2 \cdot ... \cdot a_k a_{k+1}$.
  Then by Euclid's Lemma, at least one of $a_1 a_2 \cdot ... \cdot a_k$, a_{k+1} is divisible by p.

  * (Case 1). Suppose a_{k+1} is divisible by p. Then at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$, namely a_{k+1}, is divisible by p.
  * (Case 2). Suppose $a_1 a_2 \cdot ... \cdot a_k$ is divisible by p. Then, by $P(k)$, at least one of $a_1, a_2, ..., a_k$ is divisible by p. Therefore at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$ is divisible by p.

  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.