1. Recall the notion of *predicate*:

   A **predicate with variables** $x, y, z, \cdots$ *is a statement 'modulo' the ambiguity of possibly one or several variables* $x, y, z, \cdots$. Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

   Also recall the statement (UPMI) (together with the logically equivalent statement (VPMI)) from the handout *Argument by mathematical induction*:

   (First) Principle of Mathematical Induction (in its 'usual' formulation). (UPMI).
   Let $P(n)$ be a predicate with variable n.
   Suppose the statement $P(0)$ is true.
   Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.
   Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.

   Principle of Mathematical Induction, (variant of its 'usual' formulation). (VPMI).
   Let $R(n)$ be a predicate with variable n. Let M be an integer.
   Suppose the statement $R(M)$ is true.
   Further suppose that for any $k \in [\![M, +\infty)$, if the statement $R(k)$ is true then the statement $R(k+1)$ is true.
   Then the statement $R(n)$ is true for any $n \in [\![M, +\infty)$.

   Here we give some examples on argument by mathematical induction. As explained in the handout *Argument by mathematical induction*, each of them has to follow a certain format, as dictated by the role played by the statement (UPMI) (or the statement (VPMI)) in such an argument.

2. **Example (A).**

We want to verify the statement

$$(\star)\ 0^3 + 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \text{ for any } n \in \mathbb{N}.$$

Observation. Note that $(\star)$ is of the form

'*for any* $n \in \mathbb{N}$, $P(n)$ *is true*',

in which $P(n)$ is the predicate with variable n that reads:

$$`0^3 + 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}\text{'}$$

So it makes sense to attempt to argue for $(\star)$ by mathematical induction.

Justification of $(\star)$ (by mathematical induction).

This is a predicate with variable n.

- For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition $0^3 + 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Initial step argument.

- $0^3 = 0 = \dfrac{0^2(0+1)^2}{4}$. Then $P(0)$ is true.

Here we state the induction assumption.

Induction argument.

- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $0^3 + 1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.

  We prove that $P(k+1)$ is true:

  This is done under the assumption '$P(k)$ is true'.

  We have

$$0^3+1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2[k^2+4(k+1)]}{4}$$
$$= \frac{(k+1)^2[(k+1)+1]^2}{4}.$$

  This is where $P(k)$ is used.

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

Having verified that our specific $P(n)$ satisfies the assumption in the Principle of Mathematical Induction, ... we declare that as a consequence the conclusion in the Principle of Mathematical Induction holds for our $P(n)$ as well.

3. **Example (B).**

We want to verify the statement ($\star$):

($\star$) *$n^3 - n$ is divisible by 3 for any $n \in \mathbb{N}$.*

Observation. Note that ($\star$) is of the form

‘*for any $n \in \mathbb{N}$, $P(n)$ is true*’,

in which $P(n)$ is the predicate with variable n that reads:

‘*$n^3 - n$ is divisible by 3*’.

So it makes sense to attempt to argue for ($\star$) by mathematical induction.

Justification of $(\star)$ (by mathematical induction).

- For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition '$n^3 - n$ is divisible by 3'.
- [Ask: Is $P(0)$ true?]

  $0^3 - 0 = 0 = 0 \cdot 3$

  $0 \in \mathbb{Z}$.

  Then $0^3 - 0$ is divisible by 3.

  Therefore $P(0)$ is true.

- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

  Then $k^3 - k$ is divisible by 3.

  [Ask: Is $P(k+1)$ true under the assumption that $P(k)$ is true, as stated above?]

  We prove that $P(k+1)$ is true:

  * By definition, there exits some $q \in \mathbb{Z}$ such that $k^3 - k = 3q$.

  We have

  $(k+1)^3 - (k+1) = \ldots = (k^3 - k) + 3k^2 + 3k = 3(q + k^2 + k)$.

  Also note that $q + k^2 + k \in \mathbb{Z}$. [Why? Detail?]

  Then $(k+1)^3 - (k+1)$ is divisible by 3.

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$. □

4. **Example (C).**

We want to verify the statement

$$(\star)\ \frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1} \textit{ for any } n \in \mathbb{N}\backslash\{0,1\}.$$

Observation. Note that $(\star)$ is of the form

'*for any integer n no less than 2, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

$$\text{`}\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}\text{'}.$$

So it makes sense to attempt to argue for $(\star)$ by mathematical induction.

Justification of ($\star$) (by mathematical induction).

- For any $n \in \mathbb{N} \backslash \{0, 1\}$, denote by $P(n)$ the proposition

$$\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}.$$

- Note that

$$\frac{(2 \cdot 2)!}{(2!)^2} = \frac{24}{4} = 6 > \frac{16}{3} = \frac{4^2}{2+1}.$$

Then $P(2)$ is true.

- Let $k \in \mathbb{N} \backslash \{0, 1\}$. Suppose $P(k)$ is true. Then $\dfrac{(2k)!}{(k!)^2} > \dfrac{4^k}{k+1}$.

[Roughwork.
Ask: How to deduce $P(k+1)$ from $P(k)$?
Further ask: Any 'friendly' reformulation of $P(k)$, $P(k+1)$?
Answer: $P(k)$ is the same as $\dfrac{(2k)!}{(k!)^2} \cdot \dfrac{k+1}{4^k} > 1$.
$P(k+1)$ is the same as ... ???]

We prove that $P(k+1)$ is true:

$$\frac{[2(k+1)]!}{[(k+1)!]^2} \cdot \frac{(k+1)+1}{4^{k+1}}$$

[Try to deduce that it is greater than 1.]

$$= \frac{(2k+2)(2k+1)}{(k+1)^2} \cdot \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{k+2}{4(k+1)}$$

$$= \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{(2k+1)(k+2)}{2(k+1)^2}$$

$$> 1 \cdot \frac{(2k+1)(k+2)}{2(k+1)^2} = \frac{2k^2+5k+2}{2k^2+4k+2} > 1.$$

(why?)

Then $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > \dfrac{4^{k+1}}{(k+1)+1}$.

Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N} \backslash \{0, 1\}$.

5. **Example (D)**.

We want to verify the statement

($\star$) *For any* $n \in [\![8, +\infty)$, *there exist some* $u, v \in \mathbb{N}$ *such that* $n = 3u + 5v$.

Observation. Note that ($\star$) is of the form

'*for any integer* n *no less than* 8, $P(n)$ *is true*',

in which $P(n)$ is the predicate with variable n that reads:

'*there exist some* $u, v \in \mathbb{N}$ *(dependent on* n*) such that* $n = 3u + 5v$'.

So it makes sense to attempt to argue for ($\star$) by mathematical induction.

Justification of $(\star)$ (by mathematical induction).

- For any $n \in [\![8, +\infty)$, denote by $P(n)$ the proposition that there exist some $u, v \in \mathbb{N}$ such that $n = 3u + 5v$.

- Note that $1 \in \mathbb{N}$ and $8 = 3 \cdot 1 + 5 \cdot 1$. Then $P(8)$ is true.

- Let $k \in [\![8, +\infty)$. Suppose $P(k)$ is true. Then there exist some $u, v \in \mathbb{N}$ such that $k = 3u + 5v$.

[We hope to deduce $P(k+1)$: There exist some $s, t \in \mathbb{N}$ such that $k+1 = 3s + 5t$. Ask: What to do to attain this objective?]

[Roughwork. Ask: Is the equality below possible?

$k+1 = 3(s) + 5(t)$ ← Expressions of the form $au + bv + c$ with $a, b, c \in \mathbb{Z}$.

Try: $k+1 = 3u + 5v + 1 = \ldots = 3(u+2) + 5(v-1)$

Is it 'good'? Yes when $v \geq 1$.

But what if $v = 0$?

When $v = 0$, $k = 3u$. Then $u \geq 3$, and $k+1 = 3u+1 = \ldots = 3(u-3) + 5 \cdot 2$.]

Note that $v = 0$ or $v \geq 1$.

* (Case 1). Suppose $v = 0$. Then $k = 3u$.

Since $k \geq 8$, we have $u \geq 3$. Then $u - 3 \in \mathbb{N}$.

Now $k+1 = 3u+1 = 3(u-3) + 5 \cdot 2$. (Note that $2 \in \mathbb{N}$.)

* (Case 2). Suppose $v \geq 1$. Then $v - 1 \in \mathbb{N}$.

Since $u \in \mathbb{N}$, we have $u + 2 \in \mathbb{N}$.

Now $k+1 = 3u + 5v + 1 = 3(u+2) + 5(v-1)$.

Hence in any case $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in [\![8, +\infty)$. □

6. **Example (E).**

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Suppose* α *is a number, not equal to* 1. *Then* $\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}$ *for each positive integer* n.

Observation. Note that $(\star\star)$ is of the form

'*Suppose blah-blah-blah. Then for any positive integer* n, $P(n)$ *is true*',

in which $P(n)$ is the predicate with variable n that reads:

'$\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}$'.

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction. However, the process of the argument is conducted under the 'overarching assumption'

' α *is a number, not equal to* 1'.

Justification of $(\star\star)$ (by mathematical induction).

Suppose α is a number, not equal to 1.

- For any positive integer n, denote by $P(n)$ the proposition

$$\sum_{k=1}^{n} \alpha^{k-1} = \frac{1-\alpha^n}{1-\alpha}.$$

- We have $\displaystyle\sum_{k=1}^{1} \alpha^{k-1} = 1 = \frac{1-\alpha^1}{1-\alpha}$. Hence $P(1)$ is true.

- Let $m \in \mathbb{N}\backslash\{0\}$. Suppose $P(m)$ is true. We deduce that $P(m+1)$ is true:

  We have

$$\begin{aligned}\sum_{k=1}^{m+1} \alpha^{k-1} &= \sum_{k=1}^{m} \alpha^{k-1} + \alpha^m \\ &= \frac{1-\alpha^m}{1-\alpha} + \alpha^m = \frac{1-\alpha^m}{1-\alpha} + \frac{\alpha^m - \alpha^{m+1}}{1-\alpha} = \frac{1-\alpha^{m+1}}{1-\alpha}\end{aligned}$$

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

7. **Example (F).**

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Let* $\{a_n\}_{n=1}^{\infty}$ *be the infinite sequence of real numbers defined by*

$$\begin{cases} a_1 & = 0 \\ a_{n+1} & = 2n - a_n \text{ if } n \geq 1 \end{cases}.$$

Then $a_n = n + \dfrac{(-1)^n - 1}{2}$ *for each positive integer* n.

Observation. Note that $(\star\star)$ is of the form

'*Suppose blah-blah-blah. Then for any positive integer* n, $P(n)$ *is true*',

in which $P(n)$ is the predicate with variable n that reads:

'$a_n = n + \dfrac{(-1)^n - 1}{2}$'.

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction. However, the process of the argument is conducted under the 'overarching assumption'

' $\{a_n\}_{n=1}^{\infty}$ *is the infinite sequence of real numbers defined by so-and-so.*'

Justification of (⋆⋆) (by mathematical induction).

Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence of real numbers defined by

$$\begin{cases} a_1 &= 0 \\ a_{n+1} &= 2n - a_n \text{ if } n \geq 1 \end{cases}.$$

- For any positive integer n, denote by $P(n)$ the proposition $a_n = n + \dfrac{(-1)^n - 1}{2}$.
- Note that $a_1 = 0 = 1 + \dfrac{(-1)^1 - 1}{2}$. Then $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $a_k = k + \dfrac{(-1)^k - 1}{2}$.

  We verify that $P(k+1)$ is true:

  We have

  $$a_{k+1} = 2k - a_k = 2k - \left[k + \frac{(-1)^k - 1}{2}\right]$$
  $$= (k+1) - 1 + \frac{-(-1)^k + 1}{2} = (k+1) + \frac{(-1)^{k+1} - 1}{2}$$

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

8. **Example (G).**

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Let n be an integer greater than 1. Suppose $u_1, u_2, \cdots, u_n$ are real numbers. Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.*

Remark.

Refer to the handout *Absolute Value and Triangle Inequality for the Reals*. This statement is the 'inequality part' in the Generalization of Triangle Inequality on the real line.

Observation. Note that $(\star\star)$ is of the form

'*for any integer n greater than 1, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

'*Suppose $u_1, u_2, \cdots, u_n$ are real numbers.*
Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.'

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction.

Justification of $(\star\star)$ (by mathematical induction).

- For any integer n greater than 1, denote by $P(n)$ the proposition below:

  'Suppose $u_1, u_2, \cdots, u_n$ are real numbers.
  Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.'

- $P(2)$ is an immediate consequence of the Triangle Inequality on the real line.

- Let $k \in \mathbb{N}\backslash\{0, 1\}$. Suppose $P(k)$ holds.
  We verify $P(k+1)$:

  Suppose $u_1, u_2, \cdots, u_k, u_{k+1}$ are real numbers.
  Define $v_1 = u_1$, $v_2 = u_2$, ..., $v_{k-1} = u_{k-1}$, and define $v_k = u_k + u_{k+1}$.
  By $P(k)$ and $P(2)$ in succession, we have

$$
\begin{aligned}
|u_1 + u_2 + \cdots + u_{k-1} + u_k + u_{k+1}| &= |v_1 + v_2 + \cdots + v_{k-1} + v_k| \\
&\leq |v_1| + |v_2| + \cdots + |v_{k-1}| + |v_k| \\
&= |u_1| + |u_2| + \cdots + |u_{k-1}| + |u_k + u_{k+1}| \\
&\leq |u_1| + |u_2| + \cdots + |u_{k-1}| + |u_k| + |u_{k+1}|
\end{aligned}
$$

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any integer n greater than 1.

9. **Example (H)**.

We take for granted the validity of the statement known as Euclid's Lemma:

> *Let $h, k \in \mathbb{Z}$, and p be a prime number. Suppose hk is divisible by p. Then at least one of h, k is divisible by p.*

We want to verify the statement $(\star\star)$:

$(\star\star)$ *Suppose p is a prime number. Then, for any integer n greater than 1, for any integers $a_1, a_2, \cdots, a_n$, if $a_1 a_2 \cdot \ldots \cdot a_n$ is divisible by p, then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.*

Remark. We may think of the statement $(\star\star)$ as a generalization of Euclid's Lemma to the product-of-many-integers situation.

Note that $(\star\star)$ is of the form

> '*Suppose blah-blah-blah. Then for any integer n greater than 1, $P(n)$ is true*',

in which $P(n)$ is the predicate with variable n that reads:

> '*for any integers $a_1, a_2, \cdots, a_n$, if $a_1 a_2 \cdot \ldots \cdot a_n$ is divisible by p, then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p*'.

It makes sense to attempt to argue for $(\star\star)$ by mathematical induction. However , the process of the argument is conducted under the 'overarching assumption'

> ' *p is a prime number.*'

Justification of $(\star\star)$ (by mathematical induction).

- Let p be a prime number. For any $n \in \mathbb{N}\backslash\{0,1\}$, denote by $P(n)$ the following proposition:

  Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_n$ is divisible by p. Then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.

- $P(2)$ is an immediate consequence of Euclid's Lemma.

- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.
  We prove that $P(k+1)$ is true:

  Let $a_1, a_2, \cdots, a_k, a_{k+1} \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_k a_{k+1}$ is divisible by p.
  Note that $a_1 a_2 \cdot ... \cdot a_k \in \mathbb{Z}$ and $(a_1 a_2 \cdot ... \cdot a_k) a_{k+1} = a_1 a_2 \cdot ... \cdot a_k a_{k+1}$.
  Then by Euclid's Lemma, at least one of $a_1 a_2 \cdot ... \cdot a_k$, a_{k+1} is divisible by p.

  * (Case 1). Suppose a_{k+1} is divisible by p. Then at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$, namely a_{k+1}, is divisible by p.
  * (Case 2). Suppose $a_1 a_2 \cdot ... \cdot a_k$ is divisible by p. Then, by $P(k)$, at least one of $a_1, a_2, ..., a_k$ is divisible by p. Therefore at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$ is divisible by p.

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.