MATH1050 Argument by mathematical induction.

1. **Mathematical statements and predicates**.
   - *A* **mathematical statement** *is a sentence with mathematical content (or several inter-related sentences which can be condensed into one through the appropriate use of clauses), for which it is meaningful to say it is true or it is false.*
   - *A* **predicate with variables** $x, y, z, \cdots$ *is a statement 'modulo' the ambiguity of possibly one or several variables* $x, y, z, \cdots$.

     Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.
2. Consider a statement of the form

   ($\star$) '*For any natural number n, $P(n)$ is true*'

   in which '$P(x)$' is a predicate with the variable x.

   Heuristically, the statement ($\star$) can be regarded as a concise presentation for the passage below:
   - $P(0)$ is true, and
   - $P(1)$ is true, and
   - $P(2)$ is true, and
   - $P(3)$ is true, and
   - $P(4)$ is true, and
   - $P(5)$ is true, ... *ad infinitum*.

   Mathematical induction is a method of argument which may be useful for justifying such a statement.

   This logical foundation of this method of argument is built on the validity of a statement known as the (First) Principle of Mathematical Induction.
3. **(First) Principle of Mathematical Induction (in its 'usual' formulation). (UPMI).**

   Let $P(n)$ be a predicate with variable n.

   Suppose the statement $P(0)$ is true.

   Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.

   Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.
4. **Nature of argument by mathematical induction.**

   What do we mean by giving an argument by mathematical induction for a statement, say,

   ($\star$) '*For any natural number n, $P(n)$ is true*'

   in which '$P(x)$' is a predicate with the variable x?

   To argue for ($\star$) by mathematical induction is:—

   (1) to verify that the specific $P(x)$ satisfies the assumption in the statement (UPMI), namely:

   ($\dagger$) The statement $P(0)$ is true, and

   ($\ddagger$) for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true,

   and then

   (2) to invoke the conclusion in the statement (UPMI) to conclude that $P(n)$ is true for any $n \in \mathbb{N}$.

   We can think of the process in (2) as providing a short-hand to the passage of argument below:
   - We have verified $P(0)$.
   - We have verified that if $P(0)$ is true then $P(1)$ is true.

     Since $P(0)$ is indeed true, $P(1)$ is true as well.
   - We have verified that if $P(1)$ is true then $P(2)$ is true.

     Since $P(1)$ is indeed true, $P(2)$ is true as well.
   - We have verified that if $P(2)$ is true then $P(3)$ is true.

     Since $P(2)$ is indeed true, $P(3)$ is true as well.

- We have verified that if $P(3)$ is true then $P(4)$ is true.
  Since $P(3)$ is indeed true, $P(4)$ is true as well.
- We have verified that if $P(4)$ is true then $P(5)$ is true.
  Since $P(5)$ is indeed true, $P(5)$ is true as well. .... *Ad infinitum.*

5. **Format of argument by mathematical induction.**

   By virtue of the nature of argument by mathematical induction, such an argument should proceed in a specific way.

   Suppose we indeed want to argue for the statement $(\star)$ by mathematical induction:

   $(\star)$ '*For any natural number n, $P(n)$ is true*'

   We proceed as described below:—

   - **Step (0).** Identify $P(n)$ and write it down explicitly.
   - **Step (1).** Verify the statement $P(0)$. (This is the 'initial step argument'.)
   - **Step (2).** Declare, for the sake of argument in this specific step, that we assume the statement $P(k)$ to be true for the moment. (This is called the induction assumption.) Based on such an assumption, deduce that the statement $P(k+1)$. (This is the 'induction argument'.)
   - **Step (3).** Declare that according to the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

6. **Argument by mathematical Induction, not 'starting from 0'?**

   There is nothing sacred about the number 0 in mathematical induction. The initial step in an argument by mathematical induction may be concerned with any number other than 0.

   Denote by $[\![M, +\infty)$ the set $\{n \in \mathbb{Z} : n \geq M\}$. It is the collection of all integers greater than or equal to M.

   We can prove that the statement (UPMI) is logically equivalent to the statement below:

   Principle of Mathematical Induction, (variant of its 'usual' formulation). (VPMI).

   Let $R(n)$ be a predicate with variable n. Let M be an integer.

   Suppose the statement $R(M)$ is true.

   Further suppose that for any $k \in [\![M, +\infty)$, if the statement $R(k)$ is true then the statement $R(k+1)$ is true.

   Then the statement $R(n)$ is true for any $n \in [\![M, +\infty)$.

7. When we apply (VPMI) to give an argument by mathematical induction for the statement, say,

   $(\star')$ '*For any integer n no less than (some fixed integer) M, $R(n)$ is true*'

   we proceed as described below:—

   - **Step (0).** Identify $R(n)$ and write it down explicitly.
   - **Step (1).** Verify the statement $R(M)$. (This is the 'initial step argument'.)
   - **Step (2).** Declare, for the sake of argument in this specific step, that we assume the statement $R(k)$ to be true for the moment. (This is called the induction assumption.) Based on such an assumption, deduce that the statement $R(k+1)$. (This is the 'induction argument'.)
   - **Step (3).** Declare that according to the Principle of Mathematical Induction, $R(n)$ is true for any integer n no less than M.

8. **Principle of Mathematical Induction as an axiom.**

   The Principle of Mathematical Induction is one of the few statements in mathematics which are selected as axioms: we can only choose between *believing* and *not believing* its validity.

   Or we may regard as an axiom some other statement which is logically equivalent to the Principle of Mathematical Induction. A popular choice for such a statement is the Well-ordering Principle for Integers.

9. **Definition. (Least element of a set of real numbers.)**

   Let T be a subset of $\mathbb{R}$, and $\lambda \in T$. We say that λ is a **least element** *of T if for any $x \in T$, $\lambda \leq x$.*

   Well-ordering Principle for Integers. (WOPI.)

   Let S be a subset of $\mathbb{N}$. Suppose S is non-empty. Then S has a least element.

   In plain words, the Well-ordering Principle for Integers is stating something 'intuitively so obvious' that we would have never doubted since childhood:

   > There is a smallest number in each (non-empty) collection of natural numbers.

   We can only choose between *believing* and *not believing* the validity of the Well-ordering Principle for Integers.