1. **Mathematical statements and predicates**.

- *A* **mathematical statement** *is a sentence with mathematical content (or several inter-related sentences which can be condensed into one through the appropriate use of clauses), for which it is meaningful to say it is true or it is false.*

- *A* **predicate with variables** $x, y, z, \cdots$ *is a statement 'modulo' the ambiguity of possibly one or several variables* $x, y, z, \cdots$.

  Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

2. Consider a statement of the form

(⋆) '*For any natural number* n, $P(n)$ *is true*'

in which '$P(x)$' is a predicate with the variable x.

Heuristically, the statement (⋆) can be regarded as a concise presentation for the passage below:

- $P(0)$ is true, and
- $P(1)$ is true, and
- $P(2)$ is true, and
- $P(3)$ is true, and
- $P(4)$ is true, and
- $P(5)$ is true, ... *ad infinitum.*

Mathematical induction is a method of argument which may be useful for justifying such a statement.

This logical foundation of this method of argument is built on the validity of a statement known as the (First) Principle of Mathematical Induction.

3. **(First) Principle of Mathematical Induction (in its 'usual' formulation). (UPMI).**

Let $P(n)$ be a predicate with variable n.

Suppose the statement $P(0)$ is true.

Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.

Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.

4. **Nature of argument by mathematical induction.**

What do we mean by giving an argument by mathematical induction for a statement, say,

($\star$) '*For any natural number n, $P(n)$ is true*'

in which '$P(x)$' is a predicate with the variable x?

To argue for ($\star$) by mathematical induction is:—

(1) to verify that the specific $P(x)$ satisfies the assumption in the statement (UPMI), namely:

($\dagger$) The statement $P(0)$ is true, and

($\ddagger$) for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true,

and then

(2) to invoke the conclusion in (UPMI) to conclude that $P(n)$ is true for any $n \in \mathbb{N}$.

We can think of the process in (2) as providing a short-hand to the passage of argument below:

- We have verified $P(0)$.
- We have verified that if $P(0)$ is true then $P(1)$ is true.
  Since $P(0)$ is indeed true, $P(1)$ is true as well.
- We have verified that if $P(1)$ is true then $P(2)$ is true.
  Since $P(1)$ is indeed true, $P(2)$ is true as well.
- We have verified that if $P(2)$ is true then $P(3)$ is true.
  Since $P(2)$ is indeed true, $P(3)$ is true as well.
- We have verified that if $P(3)$ is true then $P(4)$ is true.
  Since $P(3)$ is indeed true, $P(4)$ is true as well.
- We have verified that if $P(4)$ is true then $P(5)$ is true.
  Since $P(5)$ is indeed true, $P(5)$ is true as well. .... *Ad infinitum.*

5. **Format of argument by mathematical induction.**

By virtue of the nature of argument by mathematical induction, such an argument should proceed in a specific way.

Suppose we indeed want to argue for the statement $(\star)$ by mathematical induction:

$(\star)$ '*For any natural number n, $P(n)$ is true*'

We proceed as described below:—

- **Step (0).**
  Identify $P(n)$ and write it down explicitly.
- **Step (1).**
  Verify the statement $P(0)$. (This is the 'initial step argument'.)
- **Step (2).**
  Declare, for the sake of argument in this specific step, that we assume the statement $P(k)$ to be true for the moment. (This is called the induction assumption.) Based on such an assumption, deduce that the statement $P(k+1)$. (This is the 'induction argument'.)
- **Step (3).**
  Declare that according to the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

6. **Argument by mathematical Induction, not 'starting from 0'?**

There is nothing sacred about the number 0 in mathematical induction. The initial step in an argument by mathematical induction may be concerned with any number other than 0.

Denote by $[\![M, +\infty)$ the set

$$\{n \in \mathbb{Z} : n \geq M\}.$$

It is the collection of all integers greater than or equal to M.

We can prove that the statement (UPMI) is logically equivalent to the statement below:

Principle of Mathematical Induction, (variant of its 'usual' formulation). (VPMI).

Let $R(n)$ be a predicate with variable n. Let M be an integer.

Suppose the statement $R(M)$ is true.

Further suppose that for any $k \in [\![M, +\infty)$, if the statement $R(k)$ is true then the statement $R(k+1)$ is true.

Then the statement $R(n)$ is true for any $n \in [\![M, +\infty)$.

7. When we apply (VPMI) to give an argument by mathematical induction for the statement, say,

($\star'$) '*For any integer n no less than (some fixed integer) M, $R(n)$ is true*'

we proceed as described below:—

- **Step (0).**
  Identify $R(n)$ and write it down explicitly.
- **Step (1).**
  Verify the statement $R(M)$. (This is the 'initial step argument'.)
- **Step (2).**
  Declare, for the sake of argument in this specific step, that we assume the statement $R(k)$ to be true for the moment. (This is called the induction assumption.) Based on such an assumption, deduce that the statement $R(k+1)$. (This is the 'induction argument'.)
- **Step (3).**
  Declare that according to the Principle of Mathematical Induction, $R(n)$ is true for any integer n no less than M.

8. **Principle of Mathematical Induction as an axiom.**

The Principle of Mathematical Induction is one of the few statements in mathematics which are selected as axioms: we can only choose between *believing* and *not believing* its validity.

Or we may regard as an axiom some other statement which is logically equivalent to the Principle of Mathematical Induction. A popular choice for such a statement is the Well-ordering Principle for Integers.

9. **Definition. (Least element of a set of real numbers.)**

Let T be a subset of $\mathbb{R}$, and $\lambda \in T$.

We say that λ is a **least element** *of T if for any $x \in T$, $\lambda \leq x$.*

Well-ordering Principle for Integers. (WOPI.)

Let S be a subset of $\mathbb{N}$. Suppose S is non-empty. Then S has a least element.

In plain words, the Well-ordering Principle for Integers is stating something 'intuitively so obvious' that we would have never doubted since childhood:

There is a smallest number in each (non-empty) collection of natural numbers.

We can only choose between *believing* and *not believing* the validity of the Well-ordering Principle for Integers.