What is the greatest common divisor of 10000000011 and 10101?

Apply Euclidean Algorithm on the pair of numbers 10000000011, 10101:

$$\begin{array}{lllllll}
10000000011 & = & 990000 & \times & 10101 & + & 10011 \\
10101 & = & 1 & \times & 10011 & + & 90 \\
10011 & = & 111 & \times & 90 & + & 21 \\
90 & = & 4 & \times & 21 & + & 6 \\
21 & = & 3 & \times & 6 & + & 3 \\
6 & = & 2 & \times & 3 & + & 0
\end{array}$$

What is the greatest common divisor of 10000000011 and 10101?

Apply Euclidean Algorithm on the pair of numbers $a_0 = 10000000011$, $a_1 = 10101$, to obtain $a_2, a_3, a_4, a_5, a_6, a_7, \cdots$:

$$\begin{array}{lllllll}
10000000011 & = & 990000 & \times & 10101 & + & 10011 \\
a_0 & & q_1 & & a_1 & & a_2 \\
10101 & = & 1 & \times & 10011 & + & 90 \\
a_1 & & q_2 & & a_2 & & a_3 \\
10011 & = & 111 & \times & 90 & + & 21 \\
a_2 & & q_3 & & a_3 & & a_4 \\
90 & = & 4 & \times & 21 & + & 6 \\
a_3 & & q_4 & & a_4 & & a_5 \\
21 & = & 3 & \times & 6 & + & 3 \\
a_4 & & q_5 & & a_5 & & a_6 \\
6 & = & 2 & \times & 3 & + & 0 \\
a_5 & & q_6 & & a_6 & & a_7
\end{array}$$

What is the greatest common divisor of 10000000011 and 10101?

Apply Euclidean Algorithm on the pair of numbers $a_0 = 10000000011$, $a_1 = 10101$, to obtain $a_2, a_3, a_4, a_5, a_6, a_7, \cdots$:

$$\begin{aligned} a_0 &= q_1 \times a_1 + a_2 \\ a_1 &= q_2 \times a_2 + a_3 \\ a_2 &= q_3 \times a_3 + a_4 \\ a_3 &= q_4 \times a_4 + a_5 \\ a_4 &= q_5 \times a_5 + a_6 \\ a_5 &= q_6 \times a_6 \end{aligned}$$

For each $j \geq 7$, we have $a_j = 0$.

Claim: a_6 is the greatest common divisor of a_0, a_1.

What is the greatest common divisor of 10000000011 and 10101?
Apply Euclidean Algorithm on the pair of numbers $a_0 = 10000000011$, $a_1 = 10101$, to obtain $a_2, a_3, a_4, a_5, a_6, a_7, \cdots$:

$$\begin{aligned}
a_0 &= q_1 \times a_1 + a_2 \\
a_1 &= q_2 \times a_2 + a_3 \\
a_2 &= q_3 \times a_3 + a_4 \\
a_3 &= q_4 \times a_4 + a_5 \\
a_4 &= q_5 \times a_5 + a_6 \\
a_5 &= q_6 \times a_6
\end{aligned}$$

For each $j \geq 7$, we have $a_j = 0$.

Ask: Why is a_6 is a common divisor of a_0, a_1?
Answer.

$a_5 = q_6 \times a_6$.	Then a_5 is divisible by a_6.
$a_4 = q_5 \times a_5 + a_6$.	Then a_4 is divisible by a_6.
$a_3 = q_4 \times a_4 + a_5$.	Then a_3 is divisible by a_6.
$a_2 = q_3 \times a_3 + a_4$.	Then a_2 is divisible by a_6.
$a_1 = q_2 \times a_2 + a_3$.	Then a_1 is divisible by a_6.
$a_0 = q_1 \times a_1 + a_2$.	Then a_0 is divisible by a_6.

What is the greatest common divisor of 10000000011 and 10101?
Apply Euclidean Algorithm on the pair of numbers $a_0 = 10000000011$, $a_1 = 10101$, to obtain $a_2, a_3, a_4, a_5, a_6, a_7, \cdots$:

$$\begin{aligned}
a_0 &= q_1 \times a_1 + a_2 \\
a_1 &= q_2 \times a_2 + a_3 \\
a_2 &= q_3 \times a_3 + a_4 \\
a_3 &= q_4 \times a_4 + a_5 \\
a_4 &= q_5 \times a_5 + a_6 \\
a_5 &= q_6 \times a_6
\end{aligned}$$

For each $j \geq 7$, we have $a_j = 0$. We have verified that a_6 is a common divisor of a_0, a_1.

Ask: Why is a_6 the greatest common divisor of a_0, a_1?
Answer. Preparatory claim: There exist some $s, t \in \mathbb{Z}$ such that $a_6 = sa_0 + ta_1$.
Justification:

$a_6 = a_4 - q_5 \times a_5. = u_4 a_4 + v_4 a_5.$ Here $u_4 = 1$, $v_4 = -q_5$.
$a_5 = a_3 - q_4 \times a_4.$ Subst. a_5 into above. Then $a_6 = u_3 a_3 + v_3 a_4$ for some $u_3, v_3 \in \mathbb{Z}$.
$a_4 = a_2 - q_3 \times a_3.$ Subst. a_4 into above. Then $a_6 = u_2 a_2 + v_2 a_3$ for some $u_2, v_2 \in \mathbb{Z}$.
$a_3 = a_1 - q_2 \times a_2.$ Subst. a_3 into above. Then $a_6 = u_1 a_1 + v_1 a_2$ for some $u_1, v_1 \in \mathbb{Z}$.
$a_2 = a_0 - q_1 \times a_1.$ Subst. a_2 into above. Then $a_6 = sa_0 + ta_1$ for some $s, t \in \mathbb{Z}$.

What is the greatest common divisor of 10000000011 and 10101?

Apply Euclidean Algorithm on the pair of numbers $a_0 = 10000000011$, $a_1 = 10101$, to obtain $a_2, a_3, a_4, a_5, a_6, a_7, \cdots$:

$$\begin{aligned}
a_0 &= q_1 \times a_1 + a_2 \\
a_1 &= q_2 \times a_2 + a_3 \\
a_2 &= q_3 \times a_3 + a_4 \\
a_3 &= q_4 \times a_4 + a_5 \\
a_4 &= q_5 \times a_5 + a_6 \\
a_5 &= q_6 \times a_6
\end{aligned}$$

For each $j \geq 7$, we have $a_j = 0$. We have verified that a_6 is a common divisor of a_0, a_1.

Known also: There exist some $s, t \in \mathbb{Z}$ such that $a_6 = sa_0 + ta_1$.

Ask: Why is a_6 the greatest common divisor of a_0, a_1?

Answer. [Verify that for any $d \in \mathbb{Z}$, if d is a common divisor of a_0, a_1 then $|d| \leq a_6$.]

Pick any $d \in \mathbb{Z}$. Suppose d is a common divisor of a_0, a_1.

Then there exist some $s', t' \in \mathbb{Z}$ such that $a_0 = s'd$ and $a_1 = t'd$.

Now $a_6 = sa_0 + ta_1 = (ss' + tt')d$.

Note that $ss' + tt' \in \mathbb{Z}$. Since $a_6 > 0$, we have $ss' + tt' \neq 0$.

Then $a_6 = |a_6| = |ss' + tt'||d| \geq |d|$.