# MATH1050 Examples: Cauchy-Schwarz Inequality and Triangle Inequality.

1. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove the statements below:

   (a) ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 \geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$.

   (b) ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$ *iff* $a_1 = a_2 = \cdots = a_n$.

2. Let $a_1, a_2, \cdots, a_n$ be non-zero real numbers. Prove the statements below:

   (a) $n^2 \leq \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} \frac{1}{{a_k}^2}\right)$.

   (b) $n^2 = \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} \frac{1}{{a_k}^2}\right)$ *iff* $|a_1| = |a_2| = \cdots = |a_n|$.

   **Remark.** How about an argument using the Arithmetico-geometrical Inequality?

3. Prove the statements below:

   (a) *Suppose* $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n$ *are real numbers. Then*

   $$\left(\sum_{j=1}^{n} a_jb_jc_jd_j\right)^4 \leq \left(\sum_{j=1}^{n} {a_j}^4\right)\left(\sum_{j=1}^{n} {b_j}^4\right)\left(\sum_{j=1}^{n} {c_j}^4\right)\left(\sum_{j=1}^{n} {d_j}^4\right).$$

   (b) *Suppose* $r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ *are non-negative real numbers. Then*

   $$\left(\sum_{j=1}^{n} r_js_jt_j\right)^3 \leq \left(\sum_{j=1}^{n} {r_j}^3\right)\left(\sum_{j=1}^{n} {s_j}^3\right)\left(\sum_{j=1}^{n} {t_j}^3\right).$$

4. (a) Applying the Cauchy-Schwarz Inequality, or otherwise, prove the statement below (♯):

   (♯) *Suppose* $a_1, a_2, \cdots, a_n$ *are positive real numbers. and* $m$ *is a non-negative integer.*

   *Then* $\left(\sum_{j=1}^{n} {a_j}^{m+1}\right)^2 \leq \left(\sum_{j=1}^{n} {a_j}^m\right)\left(\sum_{j=1}^{n} {a_j}^{m+2}\right)$.

   (b) Applying the statement (♯), or otherwise, prove the statement (♯♯):

   (♯♯) *Let* $b_1, b_2, \cdots, b_n$ *be positive real numbers. Suppose* $\sum_{j=1}^{n} b_j = 1$.

   *Then* $\sum_{j=1}^{n} {b_j}^p \leq n \sum_{j=1}^{n} {b_j}^{p+1}$ *for each non-negative integer* $p$.

   (c) Applying the statement (♯) and/or the statement (♯♯), prove the statement (♯♯♯):

   (♯♯♯) *Suppose* $c_1, c_2, \cdots, c_n$ *are positive real numbers. Then* $\left(\sum_{j=1}^{n} c_j\right)\left(\sum_{j=1}^{n} {c_j}^r\right) \leq n \sum_{j=1}^{n} {c_j}^{r+1}$ *for each non-negative integer* $r$.

5. *In this question, you may need to apply the Cauchy-Schwarz Inequality more than once.*

   (a) Prove the statement (♯):

   (♯) *Suppose* $a_1, a_2, \cdots, a_n$ *are real numbers. Then* $\frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \leq \sum_{k=1}^{n} {a_k}^2$.

   (b) Prove the statement (♯♯):

(♯♯) *Let $b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n$ be real numbers. Further suppose that $b_1, b_2, \cdots, b_n$ are positive.*

*Then* $\left(\sum_{k=1}^{n} b_k c_k\right)^2 \leq \left(\sum_{k=1}^{n} b_k\right)\left(\sum_{j=1}^{n} b_j {c_j}^2\right).$

(c) Applying the results above, or otherwise, prove the statement (♯♯♯):

(♯♯♯) *Let $r \geq 2$. Suppose $x_1, x_2, \cdots, x_n$ are real numbers which are not all zero.*

*Then* $\left(\sum_{k=1}^{n} \frac{x_k}{r^k}\right)^2 < \sum_{k=1}^{n} \frac{{x_k}^2}{r^k}.$

6. (a) Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be non-zero real numbers.

   i. Prove the statement (♯):

   (♯) *Let $p, q$ be real numbers. Suppose $p \leq \frac{b_k}{a_k} \leq q$ for each $k = 1, 2, \cdots, n$. Then*

$$(p+q)\sum_{k=1}^{n} a_k b_k \geq \sum_{k=1}^{n} {b_k}^2 + pq\sum_{k=1}^{n} {a_k}^2.$$

   ii. Hence, or otherwise, prove the statement (♯♯):

   (♯♯) *Let $m, M$ be real numbers. Suppose $0 < m \leq a_k \leq M$ and $0 < m \leq b_k \leq M$ for each $k = 1, 2, \cdots, n$. Then*

$$\left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) \leq \frac{1}{4}\left(\frac{M}{m} + \frac{m}{M}\right)^2 \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

(b) Applying the results in the previous part, together with the Cauchy-Schwarz Inequality, or otherwise, prove the statement (♮):

(♮) *For each positive integer $n$,*

$$\left(n + \frac{1}{9}\right)^2 < \left[\sum_{k=1}^{n}\left(1 + \frac{1}{3^k}\right)^2\right]\left[\sum_{k=1}^{n}\left(1 - \frac{1}{3^{k+1}}\right)^2\right] < \frac{169}{144}\left(n + \frac{1}{3}\right)^2.$$

7. Let $p \in (1, +\infty) \cap \mathbb{Q}$. Define $q = \left(1 - \frac{1}{p}\right)^{-1}$. (Note that $q \in (1, +\infty) \cap \mathbb{Q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.)

Prove the results below:

(a) *Let $u, v$ be positive real numbers. The inequality $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$ holds.*

(b) *Let $a, b, c, d$ be positive real numbers. The inequality $ac + bd \leq (a^p + b^p)^{\frac{1}{p}}(c^q + d^q)^{\frac{1}{q}}$ holds.*

(c) *Let $w, x, y, z$ be positive real numbers. The inequality $[(w+y)^p + (x+z)^p]^{\frac{1}{p}} \leq (w^p + x^p)^{\frac{1}{p}} + (y^p + z^p)^{\frac{1}{p}}$ holds.*

**Remark.** Apply Bernoulli's Inequality in part (a). In part (b), apply the result of part (a). In part (c), apply the result of part (b). The results in part (b), part (c) are 'baby versions' of **Hölder's Inequality**, **Minkowski's Inequality** respectively. They are respectively generalizations of the Cauchy-Schwarz Inequality and the Triangle Inequality.

8. (a) Applying the Cauchy-Schwarz Inequality for real vectors, or otherwise, prove the result (†):

(†) *Suppose $\zeta_1, \zeta_2, \cdots, \zeta_n, \eta_1, \eta_2, \cdots, \eta_n$ are complex numbers.*

*Then* $\left|\sum_{j=1}^{n} \zeta_j \overline{\eta_j}\right| \leq \left(\sum_{j=1}^{n} |\zeta_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} |\eta_j|^2\right)^{\frac{1}{2}}.$

(b) Applying the statement (†), or otherwise, prove the statement (‡):

(‡) *Suppose* $\kappa_1, \kappa_2, \cdots, \kappa_n, \lambda_1, \lambda_2, \cdots, \lambda_n$ *are complex numbers.*

*Then* $\left(\sum_{j=1}^{n} |\kappa_j + \lambda_j|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} |\kappa_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^{\frac{1}{2}}$.

**Remark.** The statements in part (a), part (b) are respectively the 'inequality parts' of the Cauchy-Schwarz Inequality for complex vectors and the Triangle Inequality for complex vectors, which are generalizations of the corresponding Cauchy-Schwarz Inequality for real vectors and the Triangle Inequality for real vectors. There are also corresponding necessary and sufficient conditions for equality to hold.

9. Let $\alpha \in (0, \pi)$. Applying the Cauchy-Schwarz Inequality for definite integrals, or otherwise, prove that

$$\left(\int_{\alpha}^{\pi} \frac{\sin(u)}{u} du\right)^2 \leq \frac{1}{\pi\alpha}(\pi - \alpha)\left(\frac{\pi - \alpha}{2} + \frac{1}{4}\sin(2\alpha)\right).$$

10. Let $a, b \in \mathbb{R}$, with $a < b$, and $\varphi : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $\varphi$ is continuous on $[a, b]$, and $\varphi(t) \geq 0$ for any $t \in [a, b]$.

    Applying the Cauchy-Schwarz Inequality for definite integrals, or otherwise, prove that

$$\left(\int_a^b \varphi(u)\cos(ku)du\right)^2 + \left(\int_a^b \varphi(u)\sin(ku)du\right)^2 \leq \left(\int_a^b \varphi(u)du\right)^2 \quad \text{for any } k \in \mathbb{R}.$$

11. Let $a, b \in \mathbb{R}$, with $a < b$, and $f$ be a real-valued function of one real variable which is twice-continuously differentiable on an open interval which contains the closed and bounded interval $[a, b]$ entirely. Suppose $f(a) = f(b) = 0$.

    (a) Verify that $\int_a^b f(x)f''(x)dx = -\int_a^b (f'(x))^2 dx$.

    (b) Here we suppose that $\int_a^b (f(x))^2 dx = 1$.

    i. Prove that $\int_a^b x f(x)f'(x)dx = -\frac{1}{2}$.

    ii. By applying the Cauchy-Schwarz Inequality, or otherwise, deduce that

$$\left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right) \geq \frac{1}{4}.$$

    (c) Here we no longer suppose that $\int_a^b (f(x))^2 dx = 1$. We only suppose that $f$ is not constant on $[a, b]$.

    *Take for granted that* $\int_a^b |f(x)|^2 dx > 0$.

    i. Prove that $\left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right) \geq \frac{1}{4}\left(\int_a^b (f(x))^2 dx\right)^2$.

    ii. Hence, or otherwise, prove that $\left(\int_a^b (f''(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right)^2 \geq \frac{1}{16}\left(\int_a^b (f(x))^2 dx\right)^3$.

    *Hint.* At some stage of the argument, you may need the Cauchy-Schwarz Inequality.

12. *Take for granted the validity of the result* (†) *below about definite integrals:*

    (†) *Let* $a, b$ *be real numbers, with* $a < b$, *and let* $g, h$ *be real-valued functions of one real variable whose domains contain the interval* $[a, b]$. *Suppose* $g, h$ *are continuous on* $[a, b]$. *Further suppose that* $g(x) \leq h(x)$ *for any* $x \in [a, b]$. *Then* $\int_a^b g(t)dt \leq \int_a^b h(t)dt$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose $f$ is continuously differentiable function on $\mathbb{R}$. Suppose $f(0) = 0$ and $f(1) = 0$.

(a) Prove that $f(x) = \int_0^x f'(t)dt = -\int_x^1 f'(t)dt$ for any $x \in [0, 1]$.

(b) By applying the Cauchy-Schwarz Inequality, or otherwise, prove the statements below:

i. $(f(x))^2 \leq x \int_0^{\frac{1}{2}} (f'(t))^2 dt$ for any $x \in \left[0, \frac{1}{2}\right]$.

ii. $(f(x))^2 \leq (1 - x) \int_{\frac{1}{2}}^1 (f'(t))^2 dt$ for any $x \in \left[\frac{1}{2}, 1\right]$.

(c) Hence, or otherwise, prove that $\int_0^1 (f(x))^2 dx \leq \frac{1}{8} \int_0^1 (f'(x))^2 dx$.