MATH1050 Examples: Greatest/least element and upper/lower bound.

1. (a) Let $I = [0, 9)$. (By definition, $I = \{x \in \mathbb{R} : 0 \le x < 9\}$.)
   i. Prove that I has a least element, namely, 0.
   ii. Prove that I has no greatest element.

   (b) Let $J = [0, 9) \cap \mathbb{Q}$.
   i. Prove that J has a least element, namely, 0.
   ii. Prove that J has no greatest element.

   (c) Let $K = [0, 9) \backslash \mathbb{Q}$.
   i. Prove that K has no greatest element.
   ii. Prove that K has no least element.

2. Let $S = \left\{ x \;\middle|\; x = \dfrac{n+2}{n+1} \text{ for some } n \in \mathbb{N} \right\}$.

   (a) Prove that S has a greatest element.

   (b) Prove that S does not have any least element.

   (c) Prove that S is bounded below in $\mathbb{R}$.

3. Let $S = \left\{ x \in \mathbb{R} : x = \dfrac{1}{3^m} + \dfrac{1}{5^n} \text{ for some } m, n \in \mathbb{N} \right\}$.

   (a) Does S have any greatest element? Why?

   (b) Does S have any least element? Why?

   (c) Is S bounded below in $\mathbb{R}$? Why?

4. Let $S = \left\{ x \in \left(0, \dfrac{1}{8}\right) : x = \dfrac{b}{3^a} \text{ for some } a, b \in \mathbb{N} \right\}$, and $T = \left\{ y \in \mathbb{R} : y = \displaystyle\sum_{k=1}^{n} \frac{1}{9^k} \text{ for some } n \in \mathbb{N} \backslash \{0\} \right\}$.

   (a) Verify that $T \subset S$.

   (b) Does T have a least element? Justify your answer.

   (c) Prove that $S \not\subset T$.

   Remark. The result you obtain in part (b) may be useful.

   (d) Prove the statement ($\sharp$):

   ($\sharp$) *For any $u, v \in S$, if $u < v$ then there exists some $w \in S$ such that $u < w < v$.*

5. *Take for granted the validity of the statement below, known as* **Archimedean Principle***:*

   For any $\varepsilon \in \mathbb{R}$, if $\varepsilon > 0$ then there exists some $N \backslash \{0\}$ such that $N\varepsilon > 1$.

   Also take for granted the validity of the result ($\sharp$):

   ($\sharp$) *For any $t \in (0, +\infty)$, $0 < \sin(t) < t$.*

   Let

$$S = \left\{ x \;\middle|\; x = \sin\left(\frac{1}{n}\right) \text{ for some } n \in \mathbb{N} \backslash \{0\} \right\}, \qquad T = \left\{ x \;\middle|\; x = \cos\left(\frac{1}{n}\right) \text{ for some } n \in \mathbb{N} \backslash \{0\} \right\}.$$

   (a) i. Verify that a lower bound of S in $\mathbb{R}$ is 0.
   ii. Define $L = \{\lambda \mid \lambda \text{ is a lower bound of } S \text{ in } \mathbb{R}.\}$.
   Prove that 0 is the greatest element of L.

   (b) i. Verify that 1 is an upper bound of T in $\mathbb{R}$.
   ii. Define $M = \{\mu \mid \mu \text{ is a upper bound of } T \text{ in } \mathbb{R}.\}$.
   Prove that 1 is a least element of T.

6. Let $S = \left\{ x \in \mathbb{R} : x > 0 \text{ and there exist some } m, n \in \mathbb{N}\backslash\{0\} \text{ such that } x = \frac{1}{m} - \frac{1}{n} \right\}$.

   (a) Prove that S does not have any greatest element.

   (b) Define $U = \{\mu \in \mathbb{R} : \mu \text{ is an upper bound of } S \text{ in } \mathbb{R}\}$.

      i. Prove that $U \neq \emptyset$.

      ii. Prove that U has a least element.

   Remark. Take for granted the validity of the Archimedean Principle.

7. Let S be a subset of $\mathbb{R}$ and $\sigma \in \mathbb{R}$. Suppose σ is an upper bound of S in $\mathbb{R}$. Prove that the two statements below are logically equivalent:

   (†) *For any $\beta \in \mathbb{R}$, if β is an upper bound of S in $\mathbb{R}$ then $\sigma \leq \beta$.*

   (‡) *For any positive real number ε, there exists some $x \in S$ such that $\sigma - \varepsilon < x$.*

   Remark. We introduce/recall the definition for the notion of **least upper bound of a set** here:

   Let S be a subset of $\mathbb{R}$ and $\sigma \in \mathbb{R}$. We say σ is a least upper bound of S in $\mathbb{R}$ if both of the statements **(LU1)**, **(LU2)** *are true:*

   (LU1) *σ is an upper bound of S in $\mathbb{R}$.*

   (LU2) *For any $\beta \in \mathbb{R}$, if β is an upper bound of S in $\mathbb{R}$ then $\sigma \leq \beta$.*

   We also introduce/recall the **Least-upper-bound Axiom** for the real number system:

   Let S be a subset of $\mathbb{R}$. Suppose S is non-empty and is bounded above in $\mathbb{R}$. Then S has a least upper bound.

   What we have established in this question is an equivalent formulation for the definition for 'least upper bound of a set': we may replace **(LU2)** by:

   (LU2') *For any positive real number ε, there exists some $x \in S$ such that $\sigma - \varepsilon < x$.*

8. (a) Let $\alpha \in \mathbb{R}$, and $A = \{x \mid x < \alpha\}$. Verify the statements below:

      i. *For any $x \in A$, $y \in \mathbb{R}$, if $y < x$ then $y \in A$.*

      ii. $A \neq \emptyset$.

      iii. $A \neq \mathbb{R}$.

      iv. *For any $x \in A$, there exists $x' \in A$ such that $x < x'$.*

   (b) Let A be a subset of $\mathbb{R}$. Suppose Conditions (C1), (C2), (C3), (C4) are all satisfied:

      (C1) *For any $x \in A$, $y \in \mathbb{R}$, if $y < x$ then $y \in A$.*

      (C2) $A \neq \emptyset$.

      (C3) $A \neq \mathbb{R}$.

      (C4) *For any $x \in A$, there exists $x' \in A$ such that $x < x'$.*

   Prove that there exists some $\alpha \in \mathbb{R}$ such that $A = \{x \mid x < \alpha\}$. (*You will need the Least-upper-bound Axiom at some stage in your argument.*)

   Remark. What we have just proved is a **characterization of half-open intervals.**

Answer.

2. (a) 2 is the greatest element of S.
   (b) S has no least element.
   (c) S is bounded below by 0 in $\mathbb{R}$.
3. (a) 2 is the greatest element of S.
   (b) S has no least element.
   (c) S is bounded below by 0 in $\mathbb{R}$.
4. (a) ——
   (b) $\dfrac{1}{9}$ is the least element of T.
   (c) ——
   (d) ——