MATH1050 Examples: Inequalities and mathematical induction.

1. Apply mathematical induction to prove the statements below.

   (a) $n^2 < 2^n$ *whenever $n$ is an integer greater than 4.*

   (b) $n^3 < 3^n$ *whenever $n$ is an integer greater than 4.*

   (c) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$ *whenever $n$ is a positive integer.*

   (d) $\dfrac{n}{2} < \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{2^n} < n$ *for any $n \in \mathbb{N}\backslash\{0, 1\}$.*

   (e) $\displaystyle\prod_{k=1}^{n}[(2k)!] > [(n+1)!]^n$ *whenever $n$ is an integer greater than 1.*

   (f) $n! < \left(\dfrac{n}{2}\right)^n$ *for any integer $n$ greater than 5.*

   (g) $\dfrac{2^{2n}}{2n} < \left(\begin{array}{c} 2n \\ n \end{array}\right) < \dfrac{2^{2n}}{4}$ *for any integer $n$ greater than 7.*

2. Apply mathematical induction to prove **Bernoulli's Inequality** in the formulation below:

   *Suppose $a \in (-1, +\infty)$. Then $(1+a)^n \geq 1 + na$ for any $n \in \mathbb{N}\backslash\{0, 1\}$.*

3. Apply mathematical induction to prove the statement ($\sharp$):

   ($\sharp$) *Suppose $a, b$ are positive real numbers. Then* $\dfrac{a^n + b^n}{2} \geq \left(\dfrac{a+b}{2}\right)^n$ *for any $n \in \mathbb{N}\backslash\{0\}$.*

4. (a) Let $u, v, x, y$ be real numbers, and $\zeta = u + vi$, $\eta = x + yi$.
   By considering the number $\zeta\overline{\eta}$, or otherwise, deduce the inequality $(ux + vy)^2 \leq (u^2 + v^2)(x^2 + y^2)$.

   (b) Apply mathematical induction, with the help of the result above where appropriate, to prove that the statement ($\sharp$):

   ($\sharp$) *Let $a, b, c$ be positive real numbers. Suppose $a^2 + b^2 = c^2$. Then $a^n + b^n < c^n$ for each integer $n \geq 3$.*

5. Apply mathematical induction to prove the statement ($\sharp$):

   ($\sharp$) *Suppose $x$ is a positive real number. Then* $\dfrac{x^n}{1 + x + x^2 + \cdots + x^{2n}} \leq \dfrac{1}{2n+1}$ *for any positive integer $n$.*

6. (a) Apply mathematical induction to prove the statement ($\sharp$):

   ($\sharp$) *Suppose $x$ is a positive real number. Then $n(x^{2n+1} + 1) + x^{2n+1} + x^{2n+2} \leq (n+1)(x^{2n+3} + 1)$ for any $n \in \mathbb{N}$.*

   (b) Hence, or otherwise, prove the statement ($\natural$):

   ($\natural$) *For any $a > 0$, for any $n \in \mathbb{N}\backslash\{0\}$, $a + a^2 + a^3 + \cdots + a^{2n} \leq n(a^{2n+1} + 1)$.*

7. Apply mathematical induction to justify each of the statements below:

   (a) *Let $n \in \mathbb{N} \in \backslash\{0, 1\}$. Suppose $z_1, z_2, \cdots, z_n$ are complex numbers. Then*

   $$\sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} \leq |z_1| + |z_2| + \cdots + |z_n|.$$

   (b) *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $\theta_1, \theta_2, \cdots, \theta_n \in (0, \pi)$.*
   *Then $|\sin(\theta_1 + \theta_2 + \cdots + \theta_n)| < \sin(\theta_1) + \sin(\theta_2) + \cdots + \sin(\theta_n)$.*

   (c) *Let $n \in \mathbb{N}\backslash\{0\}$. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then*

   $$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

(d) *Suppose $s, t \in \mathbb{Q}$, with $t > 0$, and $n \in \mathbb{N}\backslash\{0\}$. Then there exist $a, b \in \mathbb{Q}$ such that $(s + \sqrt{t})^n = a + b\sqrt{t}$.*

**Remark.** You have to think carefully which proposition is to be formulated and proved by mathematical induction. Do you 'fix' $s, t$ right at the beginning, so that your proposition handles the same $s, t$ throughout the argument? Or do you accommodate all possible '$s, t$' inside the same proposition?

8. (a) Apply mathematical induction to prove the statement ($\sharp$):

($\sharp$) *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. Further suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then*

$$n\sum_{j=1}^{n} a_j b_j - \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} b_k\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}(a_j - a_k)(b_j - b_k).$$

(b) Hence, or otherwise, prove the statement below, known as **Chebychev's Inequality**:

*Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. Further suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then*

$$\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{j=1}^{n} a_j b_j$$

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9. (a) Prove the statement ($\sharp$):

($\sharp$) *Suppose $\zeta, \eta$ are complex numbers, and $c$ is a positive real number.*
*Then $|\zeta + \eta|^2 \leq (1 + c)|\zeta|^2 + (1 + \frac{1}{c})|\eta|^2$.*

(b) Apply mathematical induction, together with the result above, to prove the statement ($\sharp\sharp$):

($\sharp\sharp$) *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $z_1, z_2, \cdots, z_n$ are complex numbers, and $a_1, a_2, \cdots, a_n$ are positive real numbers. Further suppose $\sum_{j=1}^{n} \frac{1}{a_j} = 1$. Then $\left|\sum_{j=1}^{n} z_j\right|^2 \leq \sum_{j=1}^{n} a_j |z_j|^2$.*

10. (a) Prove the statement ($\sharp$):

($\sharp$) *Suppose $x, y \in (0, 0.5]$. Then $\frac{xy}{(x + y)^2} \leq \frac{(1 - x)(1 - y)}{[(1 - x) + (1 - y)]^2}$.*

(b) Apply mathematical induction to justify the statement ($\uparrow$) below:

($\uparrow$) *Let $n \in \mathbb{N}$. Suppose $a_1, a_2, \cdots, a_{2^n} \in (0, 0.5]$. Then*

$$\frac{a_1 a_2 \cdot \ldots \cdot a_{2^n}}{(a_1 + a_2 + \cdots + a_{2^n})^{2^n}} \leq \frac{(1 - a_1)(1 - a_2) \cdot \ldots \cdot (1 - a_{2^n})}{[(1 - a_1) + (1 - a_2) + \cdots + (1 - a_{2^n})]^{2^n}}$$

.

(c) Hence, or otherwise, prove that the statement ($\Uparrow$) below is true:

($\Uparrow$) *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n \in (0, 0.5]$. Then*

$$\frac{a_1 a_2 \cdot \ldots \cdot a_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq \frac{(1 - a_1)(1 - a_2) \cdot \ldots \cdot (1 - a_n)}{[(1 - a_1) + (1 - a_2) + \cdots + (1 - a_n)]^n}$$

**Remark.** Equality holds iff $a_1 = a_2 = \cdots = a_n$. (Prove it as well.) The result (together with the 'equality condition') that we have proved is known as the **Ky Fan Inequality**.

11. Let $p$ be a positive integer.

(a) Let $x$ be a real number. Suppose $0 \leq x \leq 1$.

i. Prove that $1 + x^p \geq x^k + x^{p-k}$ for each $k = 0, 1, 2, \cdots, p$.

ii. Hence, or otherwise, deduce that $(1 + x)^p \leq 2^{p-1}(1 + x^p)$.

(b) Prove the statement (♯):

(♯) *Suppose $u, v$ are positive real numbers. Then* $\left(\frac{u+v}{2}\right)^p \leq \frac{u^p+v^p}{2}$.

(c) Apply mathematical induction to justify the statement (↑) below:

(↑) *Let $n$ be a non-negative integer. Suppose $a_1, a_2, \cdots, a_{2^n}$ are positive real numbers. Then*

$$\left(\frac{a_1+a_2+\cdots+a_{2^n}}{2^n}\right)^p \leq \frac{{a_1}^p+{a_2}^p+\cdots+{a_{2^n}}^p}{2^n}.$$

(d) Hence, or otherwise, prove that the statement (⇑) below is true:

(⇑) *Let $n$ be a positive integer. Suppose $a_1, a_2, \cdots, a_n$ be positive real numbers. Then*

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^p \leq \frac{{a_1}^p+{a_2}^p+\cdots+{a_n}^p}{n}.$$

**Remark.** Suppose $p$ is a positive integer and $a_1, a_2, \cdots, a_n$ are $n$ positive real numbers. Then the number $\sqrt[p]{\frac{{a_1}^p+{a_2}^p+\cdots+{a_n}^p}{n}}$ is called the **mean power of** $a_1, a_2, \cdots, a_n$ **of order** $p$. (When $p = 1$, this number is the arithmetic mean of $a_1, a_2, \cdots, a_n$; when $p = 2$, this number is more often called the root-square-mean of $a_1, a_2, \cdots, a_n$.) What has proved here is that the mean power of a collection of finitely many positive real numbers of order $p$ is greater than or equal to the arithmetic mean of the same collection of numbers.

12. (a) Prove the statement (♮):

(♮) *Suppose $x, y$ are positive real numbers. Then* $\sqrt{(1+x)(1+y)} \geq 1+\sqrt{xy}$.

(b) Apply mathematical induction to justify the statement (↑) below:

(↑) *Let $n \in \mathbb{N}$. Suppose $a_1, a_2, \cdots, a_{2^n}$ are positive real numbers. Then*

$$\sqrt[2^n]{(1+a_1)(1+a_2)\cdot \ldots \cdot(1+a_{2^n})} \geq 1+\sqrt[2^n]{a_1a_2\cdot \ldots \cdot a_{2^n}}$$

.

(c) Hence, or otherwise, prove that the statement (⇑) below is true:

(⇑) *Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then*

$$\sqrt[n]{(1+a_1)(1+a_2)\cdot \ldots \cdot(1+a_n)} \geq 1+\sqrt[n]{a_1a_2\cdot \ldots \cdot a_n}$$

.

(d) Hence, or otherwise, deduce the statement (♯):

(♯) *Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ are positive real numbers. Then*

$$\sqrt[n]{(u_1+v_1)(u_2+v_2)\cdot \ldots \cdot(u_n+v_n)} \geq \sqrt[n]{u_1u_2\cdot \ldots \cdot u_n}+\sqrt[n]{v_1v_2\cdot \ldots \cdot v_n}$$