MATH1050 Examples: Absolute values, Triangle Inequality and beyond.

Advice.

- All questions apart from Question (1) and Question (2) are advanced. They are intended as preview material on MATH2050.

1. We introduce the definitions below:

   - *Let $a, b \in \mathbb{R}$.*
     We define the **maximum** *of a, b, which we denote by $\max(a, b)$, by*

$$\max(a,b) = \begin{cases} b & \text{if} \quad a \leq b \\ a & \text{if} \quad a > b \end{cases}$$

     We define the **minimum** *of a, b, which we denote by $\min(a, b)$, by*

$$\min(a,b) = \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$$

   Prove the statements below:

   (a) *Suppose $a, b \in \mathbb{R}$. Then $\min(a, b) \leq a \leq \max(a, b)$ and $\min(a, b) \leq b \leq \max(a, b)$.*

   (b) *Suppose $a, b \in \mathbb{R}$. Then $\max(a, b) = \dfrac{a+b+|a-b|}{2}$ and $\min(a, b) = \dfrac{a+b-|a-b|}{2}$.*

   (c) *Suppose $a, b \in \mathbb{R}$. Then $a + b = \max(a, b) + \min(b, a)$ and $|a - b| = \max(a, b) - \min(a, b)$.*

   (d) *Suppose $a, b \in \mathbb{R}$. Then $\max(a, b) = \max(b, a)$ and $\min(a, b) = \min(b, a)$.*

   (e) *Suppose $a, b \in \mathbb{R}$. Then $\max(-a, -b) = -\min(a, b)$ and $\min(-a, -b) = -\max(a, b)$.*

   (f) *Suppose $a, b, c \in \mathbb{R}$. Then $\max(\max(a, b), c) = \max(a, \max(b, c))$ and $\min(\min(a, b), c) = \min(a, \min(b, c))$.*

   (g) *Suppose $a, b, c \in \mathbb{R}$. Then $\min(\max(a, b), \max(b, c), \max(a, c)) = \max(\min(a, b), \min(b, c), \min(c, a))$.*

2. (a) Prove the statements below:

      i. *Suppose $a \in \mathbb{R}$. Then $|a| = |-a|$.*

      ii. *Suppose $b, c \in \mathbb{R}$. Then $-b \leq c \leq b$ iff $|c| \leq b$.*

   (b) Apply the results above to prove the statement below:

      - *Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.*

3. Let ε be a positive real number. Let x, y, x_0, y_0 be real numbers. Prove the statements below:

   (a) *Suppose $|x - x_0| < \dfrac{\varepsilon}{2}$ and $|y - y_0| < \dfrac{\varepsilon}{2}$. Then $|(x + y) - (x_0 + y_0)| < \varepsilon$ and $|(x - y) + (x_0 - y_0)| < \varepsilon$.*

   (b) *Suppose $|x - x_0| < \min(\dfrac{\varepsilon}{2|y_0| + 1}, 1)$ and $|y - y_0| < \dfrac{\varepsilon}{2|x_0| + 1}$. Then $|xy - x_0y_0| < \varepsilon$.*

   (c) *Suppose $y_0 \neq 0$ and $|y - y_0| < \min(\dfrac{|y_0|}{2}, \dfrac{\varepsilon|y_0|^2}{2})$. Then $y \neq 0$ and $\left|\dfrac{1}{y} - \dfrac{1}{y_0}\right| < \varepsilon$.*

4. Let c, ε be positive real numbers. Define $\delta = \min(1, \dfrac{\varepsilon}{1 + 3c + 3c^2})$.

   (a) Prove that $\delta > 0$ and $\delta \leq 1$.

   (b) Let x be a real number. Suppose $|x - c| < \delta$.

      i. Prove that $|x^2 + cx + c^2| \leq 1 + 3c + 3c^2$.

      ii. Hence, or otherwise, deduce that $|x^3 - c^3| < \varepsilon$.

Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(1, \dfrac{\varepsilon}{1+3c+3c^2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|x^3 - c^3| < \varepsilon$.* Hence we have argued for the continuity of the function t^3 at every positive value of t.

5. Let c, ε be positive real numbers. Define $\delta = \min(\dfrac{\varepsilon c^2}{2}, \dfrac{c}{2})$.

   (a) Prove that $\delta > 0$ and $\delta \leq \dfrac{c^2}{2}$.

   (b) Let x be a real number. Suppose $|x - c| < \delta$.

   i. Prove that $x > \dfrac{c}{2}$.

   ii. Hence, or otherwise, deduce that $\left|\dfrac{1}{x} - \dfrac{1}{c}\right| < \varepsilon$.

   Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\dfrac{\varepsilon c^2}{2}, \dfrac{c}{2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|1/x - 1/c| < \varepsilon$.* Hence we have argued for the continuity of the function $1/t$ at every positive value of t.

6. Let c, ε be positive real numbers. Define $\delta = \min(\varepsilon\sqrt{c}, \dfrac{c}{2})$.

   (a) Prove that $\delta > 0$ and $\delta \leq \dfrac{c^2}{2}$.

   (b) Suppose $|x - c| < \delta$.

   i. Prove that $x > \dfrac{c}{2}$.

   ii. Hence, or otherwise, deduce that $|\sqrt{x} - \sqrt{c}| < \varepsilon$.

   Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\varepsilon\sqrt{c}, \dfrac{c}{2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|\sqrt{x} - \sqrt{c}| < \varepsilon$.* Hence we have argued for the continuity of the function $\sqrt{t}$ at every positive value of t.

7. (a) Prove the statement below:

   - *Suppose $u, v \in \mathbb{R}$. Then $|u| \leq \sqrt{u^2+v^2}$ and $|u+v| \leq 2\sqrt{u^2+v^2}$.*

   (b) Let a, b be real numbers, and ε be a positive real number. Define $\delta = \dfrac{\varepsilon}{2}$.

   Suppose $\sqrt{(x-a)^2+(y-b)^2} < \delta$. Prove that $|(x+y)-(a+b)| < \varepsilon$.

   Remark. This is what we have verified overall: *For any $a, b \in \mathbb{R}$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \dfrac{\varepsilon}{2}$) such that for any $x, y \in \mathbb{R}$, if $\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|(x+y)-(a+b)| < \varepsilon$.* Hence we have argued for the continuity of the function $s+t$ at every point (s,t) on the plane $\mathbb{R}^2$.

   (c) Let a, b be real numbers, and ε be a positive real number. Define $\delta = \min(\dfrac{\varepsilon}{|a|+|b|+1}, 1)$.

   Suppose $\sqrt{(x-a)^2+(y-b)^2} < \delta$. Prove that $|xy - ab| < \varepsilon$.

   Remark. This is what we have verified overall: *For any $a, b \in \mathbb{R}$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\dfrac{\varepsilon}{|a|+|b|+1}, 1)$) such that for any $x, y \in \mathbb{R}$, if $\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|xy - ab| < \varepsilon$.* Hence we have argued for the continuity of the function $s+t$ at every point (s,t) on the plane $\mathbb{R}^2$.