1. **Well-ordering Principle for the integers and Least-upper-bound Axiom for the reals.**

   Here we take for granted the validity of two statements, one for the natural number system, the other for the real number system:

   (a) **Well-ordering Principle for the integers (WOPI).**
   Suppose S is a non-empty subset of $\mathbb{N}$. Then S has a least element.

   (b) **Least-upper-bound Axiom for the reals (LUBA).**
   Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

   With the help of the Least-upper-bound Axiom for the reals, we are going to establish the validity of two heuristically obvious statements:

   (a) **Unboundedness of the natural number system in the reals (UNR).**
   $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

   (b) **Archimedean Principle for the reals (AP).**
   For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

   The Well-ordering Principle for integers will be used later on.

2. **Unboundedness of the natural number system in the reals (UNR).**

   $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

   Proof. [Proof-by-contradiction argument.] Suppose it were true that $\mathbb{N}$ was bounded above in $\mathbb{R}$.

   Note that $0 \in \mathbb{N}$. Then $\mathbb{N} \neq \emptyset$.

   Then, by the Least-upper-bound Axiom, $\mathbb{N}$ would have a supremum in $\mathbb{R}$. We denote this number by σ.

   [*Idea on how to proceed further.* Ask: What is wrong with the existence of such a number σ? Could σ be greater, by a definite amount, say, 0.5, than every natural number? Why (not)? If no, then in light of the presence of some natural number, say, n_0, to be less than σ by at most 0.5, what can we say about the number $n_0 + 1$? Do we have a contradiction now? Why?]

   Write $\varepsilon_0 = \frac{1}{2}$. We have $\sigma - \varepsilon_0 < \sigma$. Then $\sigma - \varepsilon_0$ would not be an upper bound of $\mathbb{N}$ in $\mathbb{R}$. There would exist some $n_0 \in \mathbb{N}$ such that $n_0 > \sigma - \varepsilon_0$.

   Since $n_0 \in \mathbb{N}$, we have $n_0 + 1 \in \mathbb{N}$. Now note that $n_0 + 1 > \sigma - \epsilon_0 + 1 = \sigma + \frac{1}{2} > \sigma$. Then σ would not be an upper bound of $\mathbb{N}$ in $\mathbb{R}$. Contradiction arises.

   Hence $\mathbb{N}$ is not bounded above in $\mathbb{R}$ in the first place.

3. **Archimedean Principle for the reals (AP).**

   For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

   Proof. Pick any $\varepsilon > 0$.

   [What do we want? Name an appropriate positive integer N which satisfies $N\varepsilon > 1$. So ask: $1 \cdot \varepsilon > 1$? $2\varepsilon > 1$? $3\varepsilon > 1$? ... Or how about $\frac{1}{\varepsilon} < 1$? $\frac{1}{\varepsilon} < 2$? $\frac{1}{\varepsilon} < 3$? ...]

   Note that $\frac{1}{\varepsilon} > 0$. By the unboundedness of $\mathbb{N}$ in $\mathbb{R}$, $\frac{1}{\varepsilon}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$. Then there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. By definition, $N \in \mathbb{N}\backslash\{0\}$ and $N\varepsilon > 1$.

   Corollary to (AP). (Re-formulations of the Archimedean Principle.)

   Each of the statements below is logically equivalent to each other

   (1) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

   (2) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $\frac{1}{N} < \varepsilon$.*

(3) *For any $K > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N > K$.*

Remark. In fact (UNR) is logically equivalent to (AP). (Proof?)

4. **Dense-ness of the rationals and irrationals in the reals.**

With the help of all the above statements, we establish the validity of some heuristically obvious statements about the rational numbers and irrational numbers.

Theorem (D1). ('Dense-ness' of positive rational numbers amongst positive real numbers.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct positive real numbers, there is at least one positive rational number.

Proof. Postponed.

Corollary (D2). ('Dense-ness' of the rationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct real numbers, there is at least one rational number. Hence there are infinitely many such rational numbers. (Why?)

Proof. Exercise. (Apply Theorem (D1) in various cases.)

Corollary (D3). ('Dense-ness' of the irrationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $u \in \mathbb{R}\backslash\mathbb{Q}$ such that $\alpha < u < \beta$.

Remark. Strictly between any two distinct real numbers, there is at least one irrational number. Hence there are infinitely many such irrational numbers. (Why?)

Proof. Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. By Corollary (D2), there exists some $s \in \mathbb{Q}$ such that $\alpha < s < \beta$. Again by Corollary (D2), there exists some $t \in \mathbb{Q}$ such that $s < t < \beta$. Now define $u = s + \dfrac{t - s}{\sqrt{2}}$. By definition, u is an irrational number. (Why?) Also, $\alpha < s < u < t < \beta$. (Why?)

The phenomena described in Corollary (2) and Corollary (3) are known as dense-ness in the reals.

Definition. (Dense-ness in the reals.)

Let D be a subset of $\mathbb{R}$. D is said to be **dense in $\mathbb{R}$** *if every open interval in $\mathbb{R}$ contains some element of D.*

Corollary (D4).

$\mathbb{Q}$ is dense in $\mathbb{R}$, and $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark. Not every 'important' subset of $\mathbb{R}$ has such a property: for instance, neither $\mathbb{N}$ nor $\mathbb{Z}$ is dense in $\mathbb{R}$.

5. **Proof of Theorem (D1).**

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$.

> [Ask. What do we want? Name an appropriate rational number which lies strictly between α and β.]
> [*Idea.* Imagine you may choose some positive integer N and then will mark on the postive half-line all the points in $\{k/N \mid k \in \mathbb{N}\}$. Which N will you choose so as to *definitely guarantee* that at least one such point, say, M/N, satisfies $\alpha < M/N < \beta$? This N needs be large, but how large? What if we want $\alpha < (M+1)/N < \beta$ as well?]

Define $\varepsilon = \beta - \alpha$. By definition, $\varepsilon > 0$.

By the Archimedean Principle, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

Define $S = \left\{ m \in \mathbb{N} : m \cdot \dfrac{1}{N} > \alpha \right\}$.

By the unboundedness of $\mathbb{N}$ in $\mathbb{R}$, there exists some $p \in \mathbb{N}$ such that $p > N\alpha$. Then $p \cdot \dfrac{1}{N} > \alpha$. Therefore $p \in S$. Hence $S \neq \emptyset$.

By the Well-ordering Principle for integers, S has a least element, which we denote by M.

Define $r = \dfrac{M}{N}$. By definition, we have $r \in \mathbb{Q}$, and $r = M \cdot \dfrac{1}{N} > \alpha$.

Also by definition, $M - 1 \notin S$. (Then $M - 1 < 0$ or ($M - 1 \geq 0$ and $(M-1) \cdot \dfrac{1}{N} \leq \alpha$). If $M - 1 < 0$, then we have $(M-1) \cdot \dfrac{1}{N} < 0 \leq \alpha$.) Therefore $(M-1) \cdot \dfrac{1}{N} \leq \alpha$ (in any case).

Now $r = \dfrac{M - 1 + 1}{N} = (M-1) \cdot \dfrac{1}{N} + \dfrac{1}{N} \leq \alpha + \dfrac{1}{N} < \alpha + (\beta - \alpha) = \beta$.