1. **Well-ordering Principle for the integers and Least-upper-bound Axiom for the reals.**

Here we take for granted the validity of these two statements:

(a) **Well-ordering Principle for the integers (WOPI).**
Let S be a non-empty subset of $\mathbb{N}$. S has a least element.

(b) **Least-upper-bound Axiom for the reals (LUBA).**
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

There is a least element of the set of all upper bounds of A in $\mathbb{R}$. Such a number is called a supremum (least upper bound) of A in $\mathbb{R}$.

With the help of the Least-upper-bound Axiom for the reals, we are going to establish the validity of two heuristically obvious statements:

(a) **Unboundedness of the natural number system in the reals (UNR).**
$\mathbb{N}$ is not bounded above in $\mathbb{R}$.

(b) **Archimedean Principle for the reals (AP).**
For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

The Well-ordering Principle for integers will be used later on.

2. **Unboundedness of the natural number system in the reals (UNR).**

$\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Proof. [Proof-by-contradiction argument.]

Suppose it were true that $\mathbb{N}$ was bounded above in $\mathbb{R}$.

Note that $0 \in \mathbb{N}$. Then $\mathbb{N} \neq \emptyset$.

Then, by the Least-upper-bound Axiom, $\mathbb{N}$ would have a supremum in $\mathbb{R}$. We denote this number by σ.

[How to proceed further?

- Ask: Is what appears in this picture allowed?

  There is some $\eta > 0$ so that:

$\mathbb{N}$ lies entirely inside $(-\infty, \sigma-\eta]$.

There is absolutely no element of $\mathbb{N}$ inside $(\sigma-\eta, \sigma]$.

- Answer. No; otherwise, we would expect $\sigma-\eta$ to be an upper bound of $\mathbb{N}$ in $\mathbb{R}$, but $\sigma-\eta$ is less than σ.
- Instead, we expect such a picture below:

Now ask: where is n_0+1? Pinpoint n_0+1.]

Take $\varepsilon_0 = \frac{1}{2}$.

$\sigma - \varepsilon_0$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

[Recall what 'β is an upper bound of S' is.

Recall how to negate this statement.]

Then there would exist some $n_0 \in \mathbb{N}$ such that $n_0 > \sigma - \varepsilon_0$.

Since $n_0 \in \mathbb{N}$, we have $n_0 + 1 \in \mathbb{N}$ also.

Note that $n_0 + 1 > \sigma - \varepsilon_0 + 1 = \sigma + \frac{1}{2} > \sigma$.

But σ was an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

Contradiction arises. $\square$

3. **Archimedean Principle for the reals (AP).**

For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N\varepsilon > 1$.

Proof. Pick any $\varepsilon > 0$.

[What do we want? Name an appropriate positive integer N which satisfies $N\varepsilon > 1$.

So ask: $1 \cdot \varepsilon > 1$? $2\varepsilon > 1$? $3\varepsilon > 1$? ... Or how about $\frac{1}{\varepsilon} < 1$? $\frac{1}{\varepsilon} < 2$? $\frac{1}{\varepsilon} < 3$? ...]

Note that $\frac{1}{\varepsilon} > 0$.

By (UNR), $\frac{1}{\varepsilon}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

Recall what 'β is an upper bound of S' is. Then recall how to negate this statement.

Then there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$.

We have $N > \frac{1}{\varepsilon} > 0$. Then $N \in \mathbb{N} \backslash \{0\}$. Moreover $N\varepsilon > \frac{1}{\varepsilon} \cdot \varepsilon = 1$. $\square$

Corollary to (AP). (Re-formulations of the Archimedean Principle.)

Each of the statements below is logically equivalent to each other

(1) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N\varepsilon > 1$.*

(2) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $\frac{1}{N} < \varepsilon$.*

(3) *For any $K > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N > K$.*

Remark. In fact (UNR) is logically equivalent to (AP). (Proof?)

4. **Dense-ness of the rationals and irrationals in the reals.**

With the help of all the above statements, we establish the validity of some heuristically obvious statements about the rational numbers and irrational numbers.

Theorem (D1). ('Dense-ness' of positive rational numbers amongst positive real numbers.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct positive real numbers, there is at least one positive rational number.

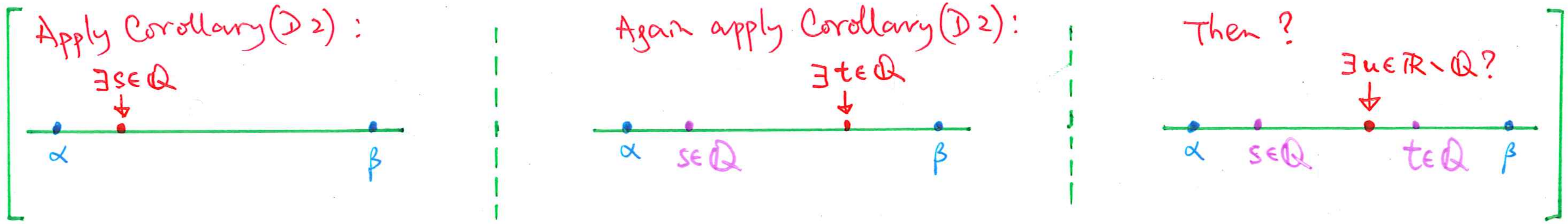
Corollary (D2). ('Dense-ness' of the rationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Corollary (D3). ('Dense-ness' of the irrationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $u \in \mathbb{R} \backslash \mathbb{Q}$ such that $\alpha < u < \beta$.

Idea in the argument?



The phenomena described in Corollary (D2) and Corollary (D3) are known as dense-ness in the reals.

Definition. (Dense-ness in the reals.)
Let D be a subset of $\mathbb{R}$.
D is said to be **dense in** $\mathbb{R}$ *if every open interval in $\mathbb{R}$ contains some element of D.*

Corollary (D4).
$\mathbb{Q}$ is dense in $\mathbb{R}$.
$\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark. Not every 'important' subset of $\mathbb{R}$ has such a property: for instance, neither $\mathbb{N}$ nor $\mathbb{Z}$ is dense in $\mathbb{R}$.

5. **Proof of Theorem (D1).**

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$.

[What do we want? Name an appropriate rational number which lies strictly between α and β.]

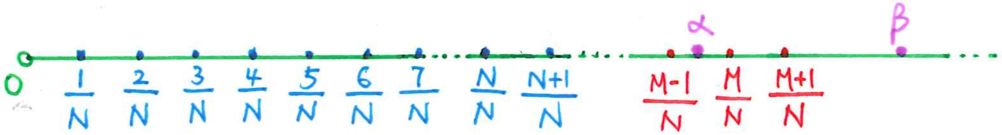
[*Idea.* Imagine you may choose some positive integer N and then will mark on the positive half-line all the points in $\{k/N \mid k \in \mathbb{N}\}$.

Which N will you choose so as to *definitely guarantee* that at least one such point, say, M/N, satisfies $\alpha < M/N < \beta$?

This N needs be large, but how large?

What if we want $\alpha < (M+1)/N < \beta$ as well?]

[Picture:



Observe: It seems that we need $\frac{1}{N} < \beta - \alpha$.]

Define $\varepsilon = \beta - \alpha$. By definition, $\varepsilon > 0$.

By (AP), there exists some $N \in \mathbb{N} \setminus \{0\}$ such that $N\varepsilon > 1$.

Define $S = \{ m \in \mathbb{N} : m \cdot \frac{1}{N} > \alpha \}$.

We have $S \neq \emptyset$. [Fill in reason: use (UNR).]

By (WOPI), S has a least element, say, M.

Now define $r = \frac{M}{N}$. By definition, $r \in \mathbb{Q}$.

[Ask: Is it true that $\alpha < r < \beta$?]

Since $r = \frac{M}{N}$ and $M \in S$, we have $r > \alpha$.

Also, by the definition of M, we have $M - 1 \notin S$.

Then $(M-1) \cdot \frac{1}{N} \leq \alpha$. [Fill in the detail.]

Therefore

$$r = \frac{M}{N} = \frac{(M-1)+1}{N} = \frac{M-1}{N} + \frac{1}{N} < \alpha + \varepsilon = \alpha + (\beta - \alpha) = \beta.$$

□