1. **What is the number e?**

You might have been told that e 'is' the limit $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

Or that e 'is' the limit $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{1}{k!}$.

But there are two questions:

(A) We are aware that some infinite sequences converge to limits and some do not. Does the infinite sequence $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}_{n=2}^{\infty}$ converge to any limit at all? Does the infinite sequence $\left\{\sum\limits_{k=0}^{n}\dfrac{1}{k!}\right\}_{n=2}^{\infty}$ converge to any limit at all?

(B) Even if both of these infinite sequences converge, do they have the same limit (which, as we have been told, is the number e)?
(The two infinite sequences do not look 'alike'; nothing suggests they have to converge to the same limit.)

This is the answer to both Question (A) and Question (B):

Theorem (1).

Let $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ be infinite sequences in $\mathbb{R}$ defined respectively by

$$a_n=\left(1+\frac{1}{n}\right)^n,\qquad b_n=\sum_{k=0}^{n}\frac{1}{k!},\qquad c_n=\left(1-\frac{1}{2n}\right)\sum_{k=0}^{n}\frac{1}{k!}\quad \text{for any } n\in\mathbb{N}\backslash\{0,1\}.$$

Then $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ converge to the same limit.

Proof of Theorem (1). Postponed; we will Lemma (2), Lemma (3) and Lemma (4) as 'stepping stones'.

Remark on notations. From now on, $\{a_n\}_{n=2}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ will refer to the same infinite sequences defined in the statement of Theorem (1).

2. **Bounded-Monotone Theorem and Sandwich Rule.**

The crucial tools used in the proof of Theorem (1) are two results that you have learnt in your *calculus* course.

Bounded-Monotone Theorem (BMT).

Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

Suppose $\{x_n\}_{n=0}^{\infty}$ is increasing. Further suppose $\{x_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$, (say, by β).

Then $\{x_n\}_{n=0}^{\infty}$ is convergent in $\mathbb{R}$. (Moreover, $\lim\limits_{n\to\infty}x_n\leq\beta$.)

Sandwich Rule (SR).

Let $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ be infinite sequences in $\mathbb{R}$.

Suppose that for any $n\in\mathbb{N}$, $u_n\leq v_n\leq w_n$. Further suppose that $\{u_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ converge to the same limit, say, ℓ in $\mathbb{R}$.

Then $\{v_n\}_{n=0}^{\infty}$ also converges to ℓ.

Remark. You will learn the proofs of these two results in your *analysis* course.

3. **Lemma (2). (Properties of $\{b_n\}_{n=2}^{\infty}$.)**

(a) *$\{b_n\}_{n=2}^{\infty}$ is strictly increasing.*

(b) *$\{b_n\}_{n=2}^{\infty}$ is bounded above by 3.*

(c) *$\lim\limits_{n\to\infty}b_n$ exists in $\mathbb{R}$, and $\lim\limits_{n\to\infty}b_n\leq 3$.*

Remark on notation. For the moment, we write $e_b=\lim\limits_{n\to\infty}b_n$.

Proof of Lemma (2).

(a) Let $n \geq 2$. We have $b_{n+1} - b_n = \dfrac{1}{(n+1)!} \geq 0$. Hence $b_{n+1} \geq b_n$.

It follows that $\{b_n\}_{n=0}^{\infty}$ is increasing.

(b) Let $n \geq 2$.

$$b_n = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \leq 1 + 1 + \sum_{k=2}^{n} \frac{1}{2^{k-1}} = 2 + \frac{1}{2} \cdot \frac{1 - 1/2^{n-1}}{1 - 1/2} = 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3$$

Therefore $\{b_n\}_{n=2}^{\infty}$ is bounded above by 3.

(c) $\{b_n\}_{n=2}^{\infty}$ is strictly increasing.

$\{b_n\}_{n=2}^{\infty}$ is also bounded above by 3.

Then by (BMT), $\{b_n\}_{n=2}^{\infty}$ converges in $\mathbb{R}$, and $\lim\limits_{n\to\infty} b_n \leq 3$.

4. **Lemma (3). (Properties of $\{a_n\}_{n=2}^{\infty}$.)**

(a) *For any* $n \in \mathbb{N}\backslash\{0, 1\}$, $a_n = 2 + \sum_{k=2}^{n} \dfrac{1}{k!} \cdot 1 \cdot \left(1 - \dfrac{1}{n}\right)\left(1 - \dfrac{2}{n}\right) \cdot ... \cdot \left(1 - \dfrac{k-1}{n}\right) < b_n$.

(b) $\{a_n\}_{n=2}^{\infty}$ *is bounded above by* 3.

(c) $\{a_n\}_{n=2}^{\infty}$ *is strictly increasing.*

(d) $\lim\limits_{n\to\infty} a_n$ *exists in* $\mathbb{R}$, *and* $\lim\limits_{n\to\infty} a_n \leq 3$.

Remark on notation. For the moment, we write $e_a = \lim\limits_{n\to\infty} a_n$.

Proof of Lemma (3).

(a) Let $n \geq 2$.

$$\begin{aligned} a_n = \left(1 + \frac{1}{n}\right)^n &= 1 + n \cdot \frac{1}{n} + \sum_{k=2}^{n} \frac{n(n-1)(n-2) \cdot ... \cdot (n-k+1)}{k!} \cdot \frac{1}{n^k} \\ &= 2 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot ... \cdot \left(1 - \frac{k-1}{n}\right) < 2 + \sum_{k=2}^{n} \frac{1}{k!} = ... = b_n \end{aligned}$$

(b) Let $n \geq 2$. We have $a_n < b_n < 3$. Therefore $\{a_n\}_{n=2}^{\infty}$ is bounded above by 3.

(c) Let $n \geq 2$. Note that $a_n > 0$ and $a_{n+1} > 0$.

$$\begin{aligned} \frac{a_{n+1}}{a_n} = ... = \left(1 + \frac{1}{n+1}\right)\left[\frac{(n+2)n}{(n+1)^2}\right]^n &= \left(1 + \frac{1}{n+1}\right)\left[1 - \frac{1}{(n+1)^2}\right]^n \\ &> \left(1 + \frac{1}{n+1}\right) \cdot \left[1 - \frac{n}{(n+1)^2}\right] \quad \text{by Bernoulli's Inequality} \\ &= ... = 1 + \frac{1}{(n+1)^3} \geq 1. \end{aligned}$$

Hence $a_{n+1} > a_n$.

It follows that $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.

(d) $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.

$\{a_n\}_{n=2}^{\infty}$ is also bounded above by 3.

Then by (BMT), $\{a_n\}_{n=2}^{\infty}$ converges in $\mathbb{R}$, and $\lim\limits_{n\to\infty} a_n \leq 3$.

5. **Lemma (4). (Properties of $\{c_n\}_{n=2}^{\infty}$.)**

(a) *For any* $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$, $c_n < a_n < b_n$.

(b) $\lim\limits_{n\to\infty} c_n$ *exists, and* $\lim\limits_{n\to\infty} c_n = e_b$.

Proof of Lemma (4).

(a) Let $n \geq 4$. We have already proved $a_n < b_n$.

$$\begin{aligned}
a_n &= 2 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) \\
&\geq 2 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \left[1 - \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{k-1}{n}\right)\right] \quad \text{by Weierstrass' Product Inequality} \\
&= 2 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \left[1 - \frac{(k-1)k}{2n}\right] \\
&= 2 + \sum_{k=2}^{n} \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^{n} \frac{(k-1)k}{k!} = 2 + \sum_{k=2}^{n} \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^{n} \frac{1}{(k-2)!} \\
&= b_n - \frac{1}{2n} b_{n-2} \\
&> b_n - \frac{1}{2n} b_n = \left(1 - \frac{1}{2n}\right) b_n = c_n
\end{aligned}$$

(b) $\lim\limits_{n\to\infty} \left(1 - \frac{1}{2n}\right)$ exists and is 1.

Also, $\lim\limits_{n\to\infty} b_n$ exists and is e_b.

Then $\lim\limits_{n\to\infty} c_n$ exists and is $1 \cdot e_b = e_b$.

6. **Proof of Theorem (1).**

By Lemma (4), for any $n \in \mathbb{N} \backslash \{0, 1, 2, 3\}$, $c_n < a_n < b_n$.

By Lemma (2), Lemma (3) and Lemma (4), the limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$, $\lim\limits_{n\to\infty} c_n$ exist. Their respective values are e_a, e_b, e_b.

Then by (SR), we have $e_b \leq e_a \leq e_b$. Hence $e_a = e_b$.

7. **Definition. (The number e.)**

We define the real number e to be the common value of the limits $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ *and* $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$.

Remark. The value of e is $2.718281828459\cdots$.

Further remark. Everything above relies on the validity of the Bounded-Monotone Theorem. The Bounded-Monotone Theorem is a consequence of the **Least-upper-bound Axiom**, which is some fundamental assumption on the nature of the real number system. (Refer to the Handout *Monotonicity and boundedness for infinite sequences of real numbers*.)

8. **Appendix 1: Beyond the number e and towards the definition of the exponential function.**

You might have been told '$\lim\limits_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$', '$\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{3^k}{k!} = e^3$', et cetera.

What '$\lim\limits_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$' telling you is :

The infinite sequence $\left\{\left(1 + \frac{2}{n}\right)^n\right\}_{n=2}^{\infty}$ *converges in* $\mathbb{R}$*, and its limit is equal to* e^2 *(which is the square of the number e as we have defined).*

But something seems to be wrong:

Even though $\lim\limits_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$ may exist, it is not immediately apparent why the equality

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n}$$

should hold.

As for '$\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{3^k}{k!}=e^3$', what it is tell you is:

The infinite sequence $\left\{\sum\limits_{k=0}^{n}\dfrac{3^k}{k!}\right\}_{n=2}^{\infty}$ *converges in* $\mathbb{R}$*, and its limit is equal to* e^3 *(which is the cube of the number* e *as we have defined).*

Again something seems to be wrong:

Even though $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{3^k}{k!}$ may exist, it is not immediately apparent why the equality

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{3^k}{k!}=\lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)^3$$

should hold.

So why are these statements true? They are the consequences of Theorem (5) and Theorem (6).

Theorem (5).

Let α *be a positive real number. Let* $\{a_n\}_{n=2}^{\infty}$*,* $\{b_n\}_{n=2}^{\infty}$*,* $\{c_n\}_{n=2}^{\infty}$ *be infinite sequences in* $\mathbb{R}$ *defined respectively by*

$$a_n=\left(1+\frac{\alpha}{n}\right)^n,\qquad b_n=\sum_{k=0}^{n}\frac{\alpha^k}{k!},\qquad c_n=\left(1-\frac{\alpha^2}{2n}\right)\sum_{k=0}^{n}\frac{\alpha^k}{k!}\quad \textit{for any } n\in\mathbb{N}\backslash\{0,1\}.$$

Then $\{a_n\}_{n=2}^{\infty}$*,* $\{b_n\}_{n=2}^{\infty}$*,* $\{c_n\}_{n=2}^{\infty}$ *converge to the same limit.*

Theorem (6).

Let σ,τ *be positive real numbers. Define* $u_n=\sum\limits_{k=0}^{n}\dfrac{\sigma^k}{k!}$*,* $v_n=\sum\limits_{k=0}^{n}\dfrac{\tau^k}{k!}$*,* $w_n=\sum\limits_{k=0}^{n}\dfrac{(\sigma+\tau)^k}{k!}$ *for each* $n\in\mathbb{N}$*. The statements below hold:*

(a) *For any* $n\in\mathbb{N}$*, the inequality* $w_n\leq u_nv_n\leq w_{2n}$ *holds.*

(b) $\lim\limits_{n\to\infty}w_n=\left(\lim\limits_{n\to\infty}u_n\right)\left(\lim\limits_{n\to\infty}v_n\right)$.

Corollary (7).

For any positive integer α*, the equality* $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{\alpha^k}{k!}=e^{\alpha}$ *holds.*

We are going to give an outline of the argument for these results. The detail will be left as a hard exercise on inequalities and limits.

(I) **Proof of Theorem (5).** First prove Lemma (5a), Lemma (5b) and Lemma (5c) as 'stepping stones'. Then imitate the argument for Theorem (1).

Remark on notations. From now on, $\{a_n\}_{n=2}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ will refer to the same infinite sequences defined in the statement of Theorem (5).

(II) **Lemma (5b). (Properties of $\{b_n\}_{n=2}^{\infty}$.)**

(a) $\{b_n\}_{n=2}^{\infty}$ *is strictly increasing.*

(b) *Suppose* N *is an integer greater than* α *and greater than* 3*. Then* $b_n\leq b_{N-1}+\dfrac{\alpha^N}{(1-\alpha/N)\cdot(N!)}$.

(c) $\{b_n\}_{n=2}^{\infty}$ *is bounded above in* $\mathbb{R}$.

(d) $\lim\limits_{n\to\infty}b_n$ *exists in* $\mathbb{R}$.

Remark on notation. For the moment, we write $E_b=\lim\limits_{n\to\infty}b_n$.

Proof of Lemma (5b). Exercise. (Imitate what has been done in the proof of Lemma (2).)

(III) **Lemma (5a). (Properties of $\{a_n\}_{n=2}^{\infty}$.)**

(a) *For any* $n \in [\![2, +\infty)$*,*

$$a_n = 1 + \alpha + \sum_{k=2}^{n} \frac{\alpha^k}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) < b_n.$$

(b) $\{a_n\}_{n=2}^{\infty}$ *is bounded above in* $\mathbb{R}$*.*

(c) $\{a_n\}_{n=2}^{\infty}$ *is strictly increasing.*

(d) $\lim\limits_{n\to\infty} a_n$ *exists in* $\mathbb{R}$*.*

Remark on notation. For the moment, we write $E_a = \lim\limits_{n\to\infty} a_n$.

Proof of Lemma (5a). Exercise. (Imitate what has been done in the proof of Lemma (3).)

(IV) **Lemma (5c). (Properties of $\{c_n\}_{n=2}^{\infty}$.)**

(a) *For any* $n \in [\![2, +\infty)$*, if* $n \geq \dfrac{\alpha^2}{2}$ *then* $c_n < a_n < b_n$*.*

(b) $\lim\limits_{n\to\infty} c_n$ *exists in* $\mathbb{R}$*, and is equal to* E_b*.*

Proof of Lemma (5c). Exercise. (Imitate what has been done in the proof of Lemma (4).)

(V) **Completion of the proof of Theorem (5).**

By Lemma (5c), for any natural number n greater than $\dfrac{\alpha^2}{2} + 2$, the inequality $c_n < a_n < b_n$ holds.

By Lemma (5a), Lemma (5b) and Lemma (5c), the limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$, $\lim\limits_{n\to\infty} c_n$ exist. Their respective values are E_a, E_b, E_b.

Then by (SR), we have $E_b \leq E_a \leq E_b$. Hence $E_a = E_b$.

(VI) Up to this point what we can say for sure is that for every positive real number α, it makes sense to talk about the limits $\lim\limits_{n\to\infty} \left(1 + \dfrac{\alpha}{n}\right)^n$ and $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{\alpha^k}{k!}$, and the limits are equal to each other.

(VII) **Proof of Theorem (6).** To verify the inequalities, 'expand' each of w_n, $u_n v_n$, w_{2n} as a sum of $\sigma^p \tau^q$, and then compare the 'expansions'. This is nothing but school algebra. For the limit result, apply (SR).

Proof of Corollary (7). Apply mathematical induction. Make use of Theorem (6).

9. **Appendix 2: From the exponential function to 'powers' and 'index laws'.**

Theorem (5) and Theorem (6) are the first steps towards making sense of the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$.

With the repeated help from the Bounded-Monotone Theorem and Sandwich Rule (and with the help of the notion of *absolute convergence for infinite series* introduced in the Handout *Cauchy-Schwarz Inequality and Triangle Inequality for square-summable sequences*), we can prove Theorem (8):

Theorem (8).

(a) *For any* $\alpha \in \mathbb{R}$*, the limits* $\lim\limits_{n\to\infty} \left(1 + \dfrac{\alpha}{n}\right)^n$, $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{\alpha^k}{k!}$ *exist in* $\mathbb{R}$ *and are equal to each other.*

(b) *For any* $\alpha, \beta \in \mathbb{R}$*, the equaltiy* $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{(\alpha+\beta)^k}{k!} = \left(\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{\alpha^k}{k!}\right)\left(\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{\beta^k}{k!}\right)$ *holds.*

Theorem (8) justifies the definition of the exponential function, and yields Theorem (9), which gives the basic (arithmetic) properties of the exponential function.

Definition. (The exponential function.)

Define the function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $\exp(x) = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{x^k}{k!}$ *for any* $x \in \mathbb{R}$*.*

exp *is called the* **exponential function** *(on the reals).*

Theorem (9).

The statements below hold:

(a) $\exp(0) = 1$, *and* $\exp(1) = e$.

(b) *For any* $s, t \in \mathbb{R}$, $\exp(s+t) = \exp(s)\exp(t)$.

(c) *For any* $s \in \mathbb{R}$, $\exp(s) > 0$ *and* $\exp(-s) = \dfrac{1}{\exp(s)}$.

Arbitrary real powers of e is in fact defined through the use of the exponential function.

Definition.

For any $\sigma \in \mathbb{R}$, *we define the number* e^σ *by* $e^\sigma = \exp(\sigma)$.

Remark. Theorem (9) immediately translates as:

(a) $e^0 = 1$, *and* $e^1 = e$.

(b) *For any* $s, t \in \mathbb{R}$, $e^{s+t} = e^s e^t$.

(c) *For any* $s \in \mathbb{R}$, $e^s > 0$ *and* $e^{-s} = \dfrac{1}{e^s}$.

You may wonder what the point of this is.

You may want to ask:

> 'Didn't we know that $e^{s+t} = e^s e^t$ for any real numbers s, t from school maths?'

The answer to this question is:

> 'In fact, in school maths we were told $e^{s+t} = e^s e^t$ for any real numbers s, t, but it was not explained why it would be so.
>
> Actually it was not explained why, for instance, $2^{\sqrt{2}+\sqrt{3}} = 2^{\sqrt{2}} \cdot 2^{\sqrt{3}}$ holds. We were not given the explanation because, in the first place, we did not know what $2^{\sqrt{2}}$ was.'

With the help of the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ and the notion of *inverse function*, we may make sense of the (natural) logarithmic function $\ln : (0, +\infty) \longrightarrow \mathbb{R}$ that we encountered in school maths. Through the exponential function and the logarithmic function we may make sense of the notion of *arbitrary real powers of arbitrary positive real numbers*, by giving an appropriate definition for them, and justify the 'index laws' for them with reference to the definition.

Definition.

Let a *be a positive real number, and* σ *be a real number. We define the number* a^σ *by* $a^\sigma = \exp(\sigma \ln(a))$.

Index Laws.

The statements below hold:

(a) *For any* $a > 0$, $a^0 = 1$ *and* $a^1 = a$.

(b) *For any* $a > 0$, *for any* $\sigma, \tau \in \mathbb{R}$, $a^{\sigma+\tau} = a^\sigma a^\tau$.

(c) *For any* $a > 0$, *for any* $\sigma \in \mathbb{R}$, $a^\sigma > 0$ *and* $a^{-\sigma} = \dfrac{1}{a^\sigma}$.

A full treatment of the above will be given in your *analysis* course.