MATH1050 Bounded-Monotone Theorem for infinite sequences.

1. We recall the definitions for the notions of *boundedness* and *monotonicity* from your *calculus of one variable* course.

   Definition. (Boundedness for infinite sequences of real numbers.)

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

   (a) Let $\kappa \in \mathbb{R}$. *We say that κ is a(n)* $\left\{ \begin{array}{c} \textbf{upper bound} \\ \textbf{lower bound} \end{array} \right\}$ **of** $\{a_n\}_{n=0}^{\infty}$ **in** $\mathbb{R}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array} \right\}$.

   (b) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{bounded above} \\ \textbf{bounded below} \end{array} \right\}$ **in** $\mathbb{R}$ *if there exists some $\kappa \in \mathbb{R}$ such that for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array} \right\}$.

   Definition. (Monotonicity and strict monotonicity for infinite sequences of real numbers.)

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

   (a) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{increasing} \\ \textbf{decreasing} \end{array} \right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq a_{n+1} \\ a_n \geq a_{n+1} \end{array} \right\}$.

   (b) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{strictly increasing} \\ \textbf{strictly decreasing} \end{array} \right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n < a_{n+1} \\ a_n > a_{n+1} \end{array} \right\}$.

   Remarks on terminology.

   (a) $\{a_n\}_{n=0}^{\infty}$ is said to be **monotonic** if $\{a_n\}_{n=0}^{\infty}$ is increasing or decreasing.

   (b) $\{a_n\}_{n=0}^{\infty}$ is said to be **strictly monotonic** if $\{a_n\}_{n=0}^{\infty}$ is strictly increasing or strictly decreasing.

2. We now recall the key result about infinite sequences below from your *calculus of one variable* course:

   Bounded-Monotone Theorem for infinite sequences of real numbers.

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

   Suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \textit{increasing} \\ \textit{decreasing} \end{array} \right\}$.

   Further suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \textit{bounded above} \\ \textit{bounded below} \end{array} \right\}$ *in $\mathbb{R}$.*

   Then $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$.

   Denote the limit of $\{a_n\}_{n=0}^{\infty}$ by σ. For any $\left\{ \begin{array}{c} \textit{upper bound} \\ \textit{lower bound} \end{array} \right\}$ *β of the infinite sequence $\{a_n\}_{n=0}^{\infty}$, the inequality* $\left\{ \begin{array}{c} \sigma \leq \beta \\ \sigma \geq \beta \end{array} \right\}$ *holds.*

3. **Example (1).**

   For any $n \in \mathbb{N}$, define $a_n = \dfrac{(n+1)(n+4)}{(n+2)(n+3)}$.

   (a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1.*

   Proof.

   Let $n \in \mathbb{N}$. We have $a_n = \dfrac{(n+1)(n+4)}{(n+2)(n+3)} = \dfrac{n^2+5n+4}{n^2+5n+6} = 1 - \dfrac{2}{n^2+5n+6} \leq 1 - 0 = 1$.

   Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by 1.

   (b) *$\{a_n\}_{n=0}^{\infty}$ is strictly increasing.*

   Proof.

   Let $n \in \mathbb{N}$.

$$\begin{aligned} a_{n+1} - a_n &= \frac{[(n+1)+1][(n+1)+4]}{[(n+1)+2][(n+1)+3]} - \frac{(n+1)(n+4)}{(n+2)(n+3)} = \frac{(n+2)(n+5)}{(n+3)(n+4)} - \frac{(n+1)(n+4)}{(n+2)(n+3)} \\ &= \frac{(n+2)^2(n+5)-(n+1)(n+4)^2}{(n+2)(n+3)(n+4)} = \frac{4}{(n+2)(n+3)(n+4)} > 0 \end{aligned}$$

   Then $a_{n+1} > a_n$.

   By the Bounded-Monotone Theorem, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Moreover, $\lim\limits_{n\to\infty} a_n \leq 1$.

   Remark. In fact, we know from 'direct calculation' that $\lim\limits_{n\to\infty} a_n = 1$.

4. **Example (2).**

   For any $n \in \mathbb{N}$, *define* $a_n = \sum_{k=0}^{n} \frac{9}{10^{k+1}}$.

   (a) $\{a_n\}_{n=0}^{\infty}$ *is bounded above in* $\mathbb{R}$ *by* 1.

   Proof.

   Let $n \in \mathbb{N}$. We have $a_n = \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10} \cdot \frac{1 - 1/10^{n+1}}{1 - 1/10} = 1 - \frac{1}{10^{n+1}} \leq 1 - 0 = 1$.

   Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by 1.

   (b) $\{a_n\}_{n=0}^{\infty}$ *is strictly increasing.*

   Proof.

   Let $n \in \mathbb{N}$. We have $a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{9}{10^{k+1}} - \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10^{n+2}} > 0$. Then $a_{n+1} > a_n$.

   Hence $\{a_n\}_{n=0}^{\infty}$ is strictly increasing.

   By the Bounded-Monotone Theorem, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Moreover, $\lim_{n \to \infty} a_n \leq 1$.

   Remark. In fact, we know from 'direct calculation' that $\lim_{n \to \infty} a_n = 1$. This equality is what what '$0.\dot{9} = 1$' really means.

5. **Example (3).**

   For any $n \in \mathbb{N} \backslash \{0\}$, *define* $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$.

   (a) $\{a_n\}_{n=1}^{\infty}$ *is bounded above in* $\mathbb{R}$ *by* 2.

   Proof.

   Let $n \in \mathbb{N} \backslash \{0\}$. We have

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 + \left(\frac{1}{1} - \frac{1}{n} \right) = 2 - \frac{1}{n} \leq 2.$$

   Hence $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

   (b) $\{a_n\}_{n=1}^{\infty}$ *is strictly increasing.*

   Proof.

   Let $n \in \mathbb{N} \backslash \{0\}$. We have $a_{n+1} - a_n = \frac{1}{(n+1)^2} > 0$. Then $a_{n+1} > a_n$.

   By the Bounded-Monotone Theorem, $\{a_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$. Moreover, $\lim_{n \to \infty} a_n \leq 2$.

   Remark. After some hard work (beyond the scope of this course), we can show that the $\{a_n\}_{n=1}^{\infty}$ converges to $\frac{\pi^2}{6}$.

   The Bounded-Monotone Theorem itself seems to say nothing about the exact value of $\lim_{n \to \infty} a_n$. (In fact this is not exactly the case.)

6. **Example (4).**

   Let p *be a positive prime number. Define* $\alpha = \sqrt{p}$. *Let* $b \in (\alpha, +\infty)$.

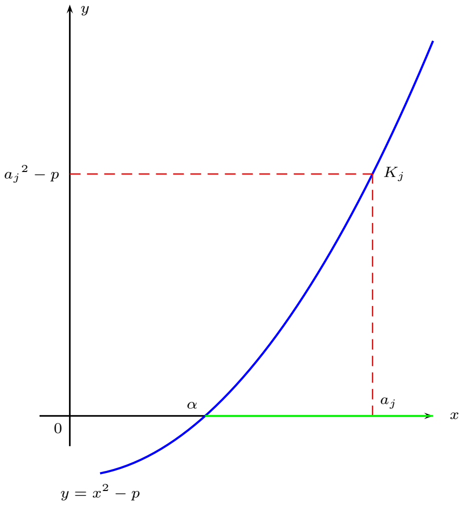
   Let $\{a_n\}_{n=0}^{\infty}$ *be the infinite sequence defined recursively by*

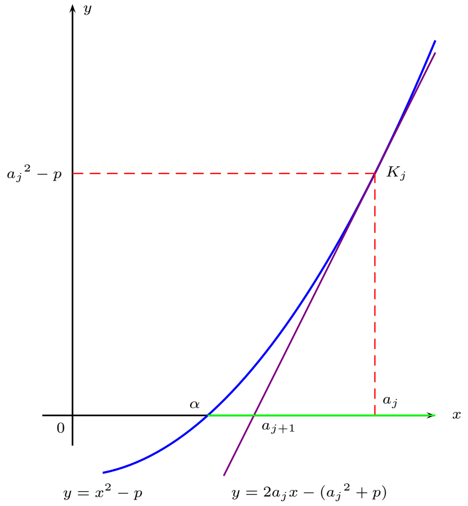
$$\begin{cases} a_0 &= b \\ a_{n+1} &= \frac{1}{2}\left(a_n + \frac{\alpha^2}{a_n} \right) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

   The infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations $a_0, a_1, a_2, a_3, \cdots$ as close to the irrational number $\alpha = \sqrt{p}$ as we like, as the index in the a_j's increases.

   It is done as described below algorithmically:

- Consider the curve $C : y = x^2 - p$ on the coordinate plane. (C intersects the x-axis at the point $(\sqrt{p}, 0)$.)
  Take $M_0 = (a_0, 0) = (b, 0)$.
  For each $j \in \mathbb{N}$, draw the line ℓ_j (whose equation is $x = a_j$) through M_j perpendicular to the x-axis. The intersection of ℓ_j with C is defined to be K_j: its coordinates are given by $K_j = (a_j, {a_j}^2 - p)$.
  Draw the tangent t_j to the curve C at K_j. (The equation of t_j is $y = 2a_jx - ({a_j}^2 + p)$.) The intersection of t_j with the x-axis is defined to be $M_{j+1} = (a_{j+1}, 0)$.





(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded below by α in $\mathbb{R}$.*
Proof.
Note that $a_0 = b > \alpha > 0$.

Suppose it were true that there existed some $n \in \mathbb{N}$ so that $a_n \leq \alpha$.
Then there would be a smallest $N \in \mathbb{N}$ so that $a_N > \alpha$ and $a_{N+1} \leq \alpha$. (Why? Apply the Well-ordering Principle for Integers.)

We would have $\alpha \geq a_{N+1} = \dfrac{1}{2}\left(a_N + \dfrac{\alpha^2}{a_N}\right) = \dfrac{{a_N}^2 + \alpha^2}{2a_N}$.

Since $a_N > 0$, we would have $2\alpha a_N \geq {a_N}^2 + \alpha^2$. Then $(a_N - \alpha)^2 = {a_N}^2 - 2\alpha a_N + \alpha^2 \leq 0$. Therefore $a_N = \alpha$.
But $a_N > \alpha$.
Contradiction arises.

Hence, in the first place, we have $a_n > \alpha$ for any $n \in \mathbb{N}$.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.*
Proof.
Let $n \in \mathbb{N}$. We have $a_n > \alpha > 0$. Then ${a_n}^2 - \alpha^2 > 0$ also.

Therefore $a_{n+1} - a_n = \dfrac{1}{2}\left(a_n + \dfrac{\alpha^2}{a_n}\right) - a_n = \dfrac{1}{2}\left(-a_n + \dfrac{\alpha^2}{a_n}\right) = -\dfrac{{a_n}^2 - \alpha^2}{2a_n} < 0$.

Then $a_{n+1} > a_n$.
Hence $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.

By the Bounded-Monotone Theorem, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Moreover, $\lim\limits_{n\to\infty} a_n \geq \alpha$.

Since $\lim\limits_{n\to\infty} a_n$ is now known to exist, we can apply the basic rules of arithmetic for limits of sequences to determine the value of $\lim\limits_{n\to\infty} a_n$:

For convenience, write $\ell = \lim\limits_{n\to\infty} a_n$. Then $\ell = \lim\limits_{n\to\infty} a_{n+1}$ also.

Since $\alpha > 0$, we have $\ell \geq \alpha > 0$. (So $1/\ell$ is well-defined.)

Recall that for any $n \in \mathbb{N}$, $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{\alpha^2}{a_n}\right)$. Then

$$\ell = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{\alpha^2}{a_n}\right) = \frac{1}{2}\left(\ell + \frac{\alpha^2}{\ell}\right).$$

Now we have $\ell^2 = \alpha^2$. Then $\ell = \alpha$ (because $\ell > 0$ and $\alpha > 0$).

Remark. Although the Bounded-Monotone Theorem does not tell us the value of $\lim\limits_{n\to\infty} a_n$, it ensures that it makes sense for us to compute it through other means. This is why this theoretical result is useful.

7. **Further digressions on Example (4).**

(1) The idea and the calculation will still work even when we do not require p to be a positive prime number; we may allow p to be any positive real number that we like. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations which descends to $\alpha = \sqrt{p}$ as close as we like, but never reaches α.

(2) How about finding cubic roots of positive real numbers?

Suppose p is a positive real number and $\alpha = \sqrt[3]{p}$. Suppose $b \in (\alpha, +\infty)$. Define infinite sequence $\{a_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} a_0 &= b \\ a_{n+1} &= \dfrac{1}{3}(2a_n + \dfrac{\alpha^3}{{a_n}^2}) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

This infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations which descends to $\alpha = \sqrt[3]{p}$ as close as we like, but never reaches α.
(First draw the picture and formulate the algorithm which are analogous to the ones for the original example on square roots. This will give you a feeling on what this infinite sequence is 'doing'. Then try to formulate and prove some statements which are analogous to the ones that we have proved for the original example.)

(3) Can you generalize the idea to finding quartic roots of positive real numbers? Quintic roots? n-th roots?

(4) The idea and method described here is a 'concrete' example of the application of **Newton's Method (for finding approximate solutions of equations)**.

8. With the help of the notion of supremum and infimum, the value of the limit guaranteed to exist in the conclusion part of the version of the *Bounded-Monontone Theorem* that you learnt in your *calculus of one variable* course can be made explicit in terms of the information provided by its assumption part.

We start by re-formulating the notion of bounded-ness for infinite sequences of real numbers in terms of bounded-ness for their corresponding 'sets of all terms'.

Lemma (1). (Boundedness for infinite sequences and boundedness for sets of all terms.)

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n \text{ for some } n \in \mathbb{N}\}$. ($T(\{a_n\}_{n=0}^{\infty})$ is the set of all terms of $\{a_n\}_{n=0}^{\infty}$.)

The statements below hold:

(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by β iff $T(\{a_n\}_{n=0}^{\infty})$ is bounded above in $\mathbb{R}$ by β.*

(b) *$\{a_n\}_{n=0}^{\infty}$ is bounded below in $\mathbb{R}$ by β iff $T(\{a_n\}_{n=0}^{\infty})$ is bounded below in $\mathbb{R}$ by β.*

Proof. Exercise. (Word game.)

9. **Bounded-Monotone Theorem for infinite sequences of real numbers, further elaborated.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Denote the set of all terms of $\{a_n\}_{n=0}^{\infty}$ by T.

Suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \text{increasing} \\ \text{decreasing} \end{array} \right\}$.

Further suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \text{bounded above} \\ \text{bounded below} \end{array} \right\}$ in $\mathbb{R}$. Denote the $\left\{ \begin{array}{c} \text{supremum} \\ \text{infimum} \end{array} \right\}$ of T in $\mathbb{R}$ by σ, if it exists.

Then $\left\{ \begin{array}{c} \sup(T) \\ \inf(T) \end{array} \right\}$ exists in $\mathbb{R}$, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, and $\lim\limits_{n\to\infty} a_n = \sigma$.

Furthermore, for any $\left\{ \begin{array}{c} \text{upper bound} \\ \text{lower bound} \end{array} \right\}$ β of the infinite sequence $\{a_n\}_{n=0}^{\infty}$, the inequality $\left\{ \begin{array}{c} \sigma \leq \beta \\ \sigma \geq \beta \end{array} \right\}$ holds. Also, for any $k \in \mathbb{N}$, the inequality $\left\{ \begin{array}{c} a_k \leq \sigma \\ a_k \geq \sigma \end{array} \right\}$ holds.

Remark. The proof of the Bounded-Monotone Theorem relies on the **Least-upper-bound Axiom**:

Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

10. An appropriate argument for the Bounded-Monotone Theorem also presumes a satisfactory definition for the notion of limit of infinite sequence of real numbers:

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers, and ℓ be a real number.
We say that $\{a_n\}_{n=0}^{\infty}$ converges to ℓ, and write $\lim\limits_{n\to\infty} a_n = \ell$ if the condition ($\star$) is satisfied:

($\star$) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $k > N$ then $|a_k - \ell| < \varepsilon$.*

11. **Proof of the Bounded-Monotone Theorem.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is increasing, and is bounded above in $\mathbb{R}$.

Define $T = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$. Note that $a_0 \in T$. Then $T \neq \emptyset$.

By assumption T is bounded above in $\mathbb{R}$. Then by the Least-upper-bound Axiom, T has a supremum in $\mathbb{R}$. Write $\sigma = \sup(T)$.

We verify that $\{a_n\}_{n=0}^{\infty}$ converges to σ, according to the definition for limit of infinite seqeunce:

- Pick any $\varepsilon > 0$.

  [Ask: Can we name an appropriate natural number N for which it happens that whenever $k > N$, $|a_k - \sigma| < \varepsilon$? How?]

  Note that $\sigma - \varepsilon < \sigma$.
  Then by definition, $\sigma - \varepsilon$ is not an upper bound of T in $\mathbb{R}$.
  Therefore there exists some $x \in T$ such that $x > \sigma - \varepsilon$.
  For the same x, there exists some $N \in \mathbb{N}$ such that $x = a_N$.

  [Observation: For such an N, we in fact have $\sigma - \varepsilon < a_N \leq \sigma < \sigma + \varepsilon$ (because σ is an upper bound of T). As a consequence $|a_N - \sigma| < \varepsilon$.
  Ask: Is it true that whenever $k > N$, $|a_k - \sigma| < \varepsilon$? Or equivalently, whenever $k > N$, $\sigma - \varepsilon < a_k < \sigma + \varepsilon$?]

  Pick any $k \in \mathbb{N}$. Suppose $k > N$.
  Then we have $a_k > a_N = x > \sigma - \varepsilon$ by assumption.
  Moreover, $\sigma - \varepsilon < a_k \leq \sigma < \sigma + \varepsilon$ (because σ is an upper bound of T). Therefore $|a_k - \sigma| < \varepsilon$.

The result follows.

12. Many infinite sequences which produce interesting limits are those which are strictly monotonic and bounded. Examples (2), (3), (4) are good illustrations. For such an infinite sequence, because of Lemma (2) (whose proof is left as an exercise), its limit is 'better and better approximated' by the terms of the sequence, but never 'attained' by any individual term.

Lemma (2).

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$. Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$. The statements below hold:

(a) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly increasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no greatest element.*

(b) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no least element.*

The conceptual role of supremum/infimum in the theoretical device *Bounded-Monotone Theorem* in pinpointing the limit for such an infinite sequence is indispensable.

It is in trying to make sense of the Bounded-Monotone Theorem and to give an 'purely algebraic justification' for this statement (in contrast to appealing to geometric intuition alone) that prompted Dedekind to inspect the real number system carefully in the first place. This is a key moment in the history of mathematics.