MATH1050 Greatest/least element, upper/lower bound

1. **Definition.**
   Let S be a subset of $\mathbb{R}$.

   (a) *Let $\lambda \in S$. λ is said to be a* $\left\{\begin{array}{c}\textbf{greatest}\\ \textbf{least}\end{array}\right\}$ **element** *of S if, for any $x \in S$,* $\left\{\begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array}\right\}$.

   (b) *S is said to* **have a** $\left\{\begin{array}{c}\textbf{greatest}\\ \textbf{least}\end{array}\right\}$ **element** *if there exists some $\lambda \in S$ such that for any $x \in S$,* $\left\{\begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array}\right\}$.

   Remarks.

   (A) Suppose S has a $\left\{\begin{array}{c}\text{greatest}\\ \text{least}\end{array}\right\}$ element. Then it is the unique $\left\{\begin{array}{c}\text{greatest}\\ \text{least}\end{array}\right\}$ element of S, in the sense below:

   * *Suppose λ, λ' are* $\left\{\begin{array}{c}\textit{greatest}\\ \textit{least}\end{array}\right\}$ *elements of S. Then $\lambda = \lambda'$.*

   We may replace the article '*a*' in 'a $\left\{\begin{array}{c}\text{greatest}\\ \text{least}\end{array}\right\}$ element' by the article '*the*'.

   (B) **Notation.** We denote the $\left\{\begin{array}{c}\text{greatest}\\ \text{least}\end{array}\right\}$ element of S by $\left\{\begin{array}{c}\max(S)\\ \min(S)\end{array}\right\}$.

   (C) **Terminology.** We may choose to write '*S has a greatest element*' as $\max(S)$ *exists*'. Et cetera. The situation is analogous for least element.

2. **Definition.**
   Let S be a subset of $\mathbb{R}$.

   (a) *Let $\beta \in \mathbb{R}$. β is said to be a(n)* $\left\{\begin{array}{c}\textbf{upper}\\ \textbf{lower}\end{array}\right\}$ **bound of** *S* **in** *$\mathbb{R}$ if, for any $x \in S$,* $\left\{\begin{array}{c} x \leq \beta \\ x \geq \beta \end{array}\right\}$.

   (b) *S is said to be* **bounded** $\left\{\begin{array}{c}\textbf{above}\\ \textbf{below}\end{array}\right\}$ **in** *$\mathbb{R}$ if there exists some $\beta \in \mathbb{R}$ such that for any $x \in S$,* $\left\{\begin{array}{c} x \leq \beta \\ x \geq \beta \end{array}\right\}$.

   (c) S is said to be **bounded in** $\mathbb{R}$ if S is bounded above in $\mathbb{R}$ and bounded below in $\mathbb{R}$.

   Remarks.

   (A) If β is an upper bound of S, then every number greater than β is also an upper bound of S. Therefore S has infinitely many upper bounds. It does not make sense to write '*the* upper bound of S is so-and-so'.
   The situation is similar for 'being bounded below'.

   (B) *Suppose λ is a greatest element of S. Then λ is an upper bound of S.*
   (How about its converse?)
   The situation is similar for 'least element' and 'lower bound'.

3. **Example (A). (Well-ordering Principle for Integers.)**

   Recall this statement below, known as the Well-ordering Principle for Integers (WOPI):

   Let S be a non-empty subset of $\mathbb{N}$. S has a least element.

   There are various re-formulations of the statement (WOPI):

   - (WOPIL) *Let T be a non-empty subset of $\mathbb{Z}$. Suppose T is bounded below in $\mathbb{R}$ by some $\beta \in \mathbb{Z}$. Then T has a least element.*
   - (WOPIG) *Let U be a non-empty subset of $\mathbb{Z}$. Suppose U is bounded above in $\mathbb{R}$ by some $\gamma \in \mathbb{Z}$. Then U has a greatest element.*

   (The proof for the logical equivalence of (WOPI), (WOPIL), (WOPIG) is left as an exercise.)

   From now on all three of them are referred to as the Well-ordering Principle for Integers.

4. **Example (B).**

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?
$[0,1)$	0 $_{Aa}$	*nil* $_{Ab}$	Yes (by 0) $_{Ac}$	Yes (by 1) $_{Ad}$
$[0,+\infty)$	0 $_{Ba}$	*nil* $_{Bb}$	Yes (by 0) $_{Bc}$	No $_{Bd}$
$(0,+\infty)$	*nil* $_{Ca}$	*nil* $_{Cb}$	Yes (by 0) $_{Cc}$	No $_{Cd}$
$[0,1)\cap\mathbb{Q}$	0 $_{Da}$	*nil* $_{Db}$	Yes (by 0) $_{Dc}$	Yes (by 1) $_{Dd}$
$[0,+\infty)\cap\mathbb{Q}$	0 $_{Ea}$	*nil* $_{Eb}$	Yes (by 0) $_{Ec}$	No $_{Ed}$
$[0,1)\backslash\mathbb{Q}$	*nil* $_{Fa}$	*nil* $_{Fb}$	Yes (by 0) $_{Fc}$	Yes (by 1) $_{Fd}$
$[0,+\infty)\backslash\mathbb{Q}$	*nil* $_{Ga}$	*nil* $_{Gb}$	Yes (by 0) $_{Gc}$	No $_{Gd}$

The detail for the argument here is relatively easy to work out, because each of the sets concerned are constructed using just one interval whose endpoints are rational numbers.

Where the set concerned has a least/greatest element, or is bounded above/below in $\mathbb{R}$, we may just name an appropriate number which serves as a least/greatest element of the set or an upper/lower bound of the set in $\mathbb{R}$, and verify that this number satisfies the condition specified in the relevant definition.

(Aa): $[0,1)$ *has a least element, namely,* 0.

Proof.
Write $S=[0,1)$. Note that $0\in S$.
Pick any $x\in S$. By definition, $0\leq x<1$. In particular $x\geq 0$.
It follows that 0 is the least element of S.

Remark. The argument for each of (Ba), (Da), (Ea) is similar.

(Ad): $[0,1)$ *is bounded above in* $\mathbb{R}$ *by* 1.

Proof.
Write $S=[0,1)$.
Pick any $x\in S$. By definition, $0\leq x<1$. In particular $x<1$. (So $x\leq 1$ also holds.)
It follows that S is bounded above by 1.

Remark. The argument for each of (Ac)-(Gc), (Dd), (Fd) is similar.

Here we focus on arguments which explain why a certain set fails to have a least/greatest element or to be bounded above/below in $\mathbb{R}$:

(Ab): $[0,1)$ *has no greatest element.*

Proof.
Write $S=[0,1)$. Suppose S had a greatest element, say, λ.
(By definition, for any $x\in S$, $x\leq\lambda$.)
Then $\lambda\in S$. Therefore $0\leq\lambda<1$.
Define $x_0=\dfrac{\lambda+1}{2}$.
Then $0\leq\lambda<x_0<1$.
Note that $x_0\in S$. But $\lambda<x_0$.
So λ would not be a greatest element of S. Contradiction arises.
Hence S has no greatest element in the first place.

Remark. The argument for each of (Ca), (Fa), (Ga), (Db), (Fb) is similar.

(Bb): $[0,+\infty)$ *has no greatest element.*

Proof.
Write $S=[0,+\infty)$. Suppose S had a greatest element, say, λ.
(By definition, for any $x\in S$, $x\leq\lambda$.)
Then $\lambda\in S$. $\lambda\geq 0$.
Define $x_0=\lambda+1$.
Then $0\leq\lambda<x_0$.

Note that $x_0 \in S$. But $\lambda < x_0$.
So λ would not be a greatest element of S. Contradiction arises.
Hence S has no greatest element in the first place.

Remark. The argument for each of (Cb), (Eb), (Gb) is similar.

(Bd): $[0, +\infty)$ *is not bounded above in* $\mathbb{R}$.

Proof.
Write $S = [0, +\infty)$. Suppose S were bounded above in $\mathbb{R}$, say, by some $\beta \in \mathbb{R}$.
(By definition, for any $x \in S$, $x \leq \beta$.)
Then, since $0 \in S$, we have $\beta \geq 0$.
Define $x_0 = \beta + 1$.
Then $0 \leq \beta < x_0$.
Note that $x_0 \in S$. But $\beta < x_0$.
So β would not be an upper bound of S in $\mathbb{R}$. Contradiction arises.
Hence S is not bounded above in $\mathbb{R}$ in the first place.

Remark. The argument for each of (Cd), (Ed), (Gd) is similar.

5. **Example (C).**

Let $S = \{x \in \mathbb{R} : x^2 \leq (\sqrt{2}+1)x - \sqrt{2}\}$, *and* $T = S \backslash \mathbb{Q}$.
(S is in fact the solution set of the inequality $x^2 \leq (\sqrt{2}+1)x - \sqrt{2}$ with unknown x in the reals.)

(a) *S has a greatest element and S has a least element.*

Proof.

- Note that $S = [1, \sqrt{2}]$.
  S has a greatest element, namely $\sqrt{2}$.
  S has a least element, namely 1.

(b) *S is bounded above and below in $\mathbb{R}$.*

Proof.

- S has a least element and a greatest element. They are respectively a lower bound and an upper bound of S in $\mathbb{R}$.

(c) *T has a greatest element, and T has no least element.*

Proof.

- Note that $T = [1, \sqrt{2}] \backslash \mathbb{Q}$.
- We have $\sqrt{2} \in [1, \sqrt{2}]$, and $\sqrt{2}$ is irrational. Then $\sqrt{2} \in T$.
  Pick any $x \in T$. By definition, $1 \leq x \leq \sqrt{2}$ and x is irrational. In particular $x \leq \sqrt{2}$.
  Therefore, by definition, $\sqrt{2}$ is a greatest element of T.
- Suppose T had a least element, say, λ. By definition, λ is irrational and $1 \leq \lambda \leq \sqrt{2}$.
  Since λ is irrational, $\lambda \neq 1$. Then $\lambda > 1$.
  Define $x_0 = \dfrac{1+\lambda}{2}$. By definition, $1 < x_0 < \lambda \leq \sqrt{2}$.
  Moreover x_0 is irrational. (Why? Fill in the detail.)
  Then $x_0 \in T$. But $x_0 < \lambda$ and λ is a least element of T. Contradiction arises.

(d) *T is bounded above and below in $\mathbb{R}$.*

Proof.

- T has a least element. It is a lower bound of T in $\mathbb{R}$.
- By definition, for any $x \in T$, $x \leq \sqrt{2}$. Then $\sqrt{2}$ is an upper bound of T in $\mathbb{R}$.

6. **Example (D).**

Let $S = \left\{ \dfrac{1}{m+1} + \dfrac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}$.

(a) *S has a greatest element and S has no least element.*

(b) *S is bounded above and below in $\mathbb{R}$.*

Proof of (a).

- Note that $2 \in S$, because $2 = \frac{1}{0+1} + \frac{1}{0+1}$ and $0 \in \mathbb{N}$.

  Pick any $x \in S$. There exists some $m, n \in \mathbb{N}$ such that $x = \frac{1}{m+1} + \frac{1}{n+1}$.

  Since $m \geq 0$ and $n \geq 0$, $x \leq \frac{1}{0+1} + \frac{1}{0+1} = 2$.

  Then 2 is a greatest element of S.
- Suppose S had a least element, say, λ.

  By definition, $\lambda \in S$. Then there exist some $m_0, n_0 \in \mathbb{N}$ such that $\lambda = \frac{1}{m_0+1} + \frac{1}{n_0+1}$.

  Take $x_0 = \frac{1}{m_0+1} + \frac{1}{n_0+2}$. By definition, $x_0 \in S$. (Why?)

  Since $0 < n_0 + 1 < n_0 + 2$, we have $x_0 = \frac{1}{m_0+1} + \frac{1}{n_0+2} < \frac{1}{m_0+1} + \frac{1}{n_0+1} = \lambda$.

  So $x_0 \in S$ and $x_0 < \lambda$. But λ was a least element of S by assumption. Contradiction arises.

7. **Example (B) re-visited for a special observation.**

   Re-consider each subset S of $\mathbb{R}$ studied in Example (B):

   (a) If S is bounded below in $\mathbb{R}$, then its lower bounds seem to form the closed interval $(-\infty, 0]$, with greatest element, 0. (This claim is easy to verify.)

   We may refer to the number 0 as the greatest amongst all lower bounds of S in $\mathbb{R}$, or simply, a greatest lower bound of S in $\mathbb{R}$.

   (b) If S is bounded above in $\mathbb{R}$, then its upper bounds seem to form a closed interval of the form $[1, +\infty)$, with least element 1.

   We may refer to the number 1 as the least amongst all upper bounds of S in $\mathbb{R}$, or simply, a least upper bound of S in $\mathbb{R}$.

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?	set of all lower bounds?	set of all upper bounds?	greatest lower bound?	least upper bound?
$[0,1)$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty,0]$	$[1,+\infty)$	0	1
$[0,+\infty)$	0	*nil*	Yes (by 0)	No	$(-\infty,0]$	$\emptyset$	0	*nil*
$(0,+\infty)$	*nil*	*nil*	Yes (by 0)	No	$(-\infty,0]$	$\emptyset$	0	*nil*
$[0,1)\cap\mathbb{Q}$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty,0]$	$[1,+\infty)$	0	1
$[0,+\infty)\cap\mathbb{Q}$	0	*nil*	Yes (by 0)	No	$(-\infty,0]$	$\emptyset$	0	*nil*
$[0,1)\backslash\mathbb{Q}$	*nil*	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty,0]$	$[1,+\infty)$	0	1
$[0,+\infty)\backslash\mathbb{Q}$	*nil*	*nil*	Yes (by 0)	No	$(-\infty,0]$	$\emptyset$	0	*nil*

   Remark. The verification of the claim ($\sharp$) is non-trivial:

   ($\sharp$) *The set of all upper bounds of the set* $[0,1) \cap \mathbb{Q}$ *is* $[1, +\infty)$.

   The other two claims are easy to verify.

8. **Example (C) re-visited for a special observation.**

   Re-consider Example (C), in which

$$S = \{x \in \mathbb{R} : (\sqrt{2}+1)x - \sqrt{2}\} = [1, \sqrt{2}], \quad T = S \backslash \mathbb{Q}.$$

   (a) The lower bounds of S form the closed interval $(-\infty, 1]$, with greatest element 1. (This claim is easy to verify.)

   We refer to the number 1 as the greatest lower bound of S.

   (b) The upper bounds of S form the closed interval $[\sqrt{2}, +\infty)$, with least element $\sqrt{2}$. (This claim is easy to verify.)

   We refer to the number $\sqrt{2}$ as the least upper bound of S.

   (c) The lower bounds of T form the closed interval of the form $(-\infty, 1]$, with greatest element 1. (This claim is easy to verify.)

   We refer to the number 1 as the greatest lower bound of T.

(d) The upper bounds of T form the closed interval of the form $[\sqrt{2}, +\infty)$, with least element $\sqrt{2}$. (The verification of this claim is non-trivial.)

We refer to the number $\sqrt{2}$ as the least upper bound of T.

9. **Example (D) re-visited for a special observation.**

Re-consider Example (D), in which

$$S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}.$$

(a) The lower bounds of S form the closed interval $(-\infty, 0]$, with greatest element 0. (The verification of this claim is non-trivial.)

We refer to the number 0 as the greatest lower bound of S.

(b) The upper bounds of S form the closed interval $[2, +\infty)$, with least element 2. (This claim is easy to verify.)

We refer to the number 2 as the least upper bound of S.

10. The phenomena discovered in the re-considerations of Examples (B), (C), (D) motivate the definitions for the notion of *supremum*, *infimum* below.

Definition.

Let S be a subset of $\mathbb{R}$, and σ be a real number.

Suppose S is $\left\{ \begin{array}{c} \textit{bounded above} \\ \textit{bounded below} \end{array} \right\}$ *in $\mathbb{R}$, and σ is a(n)* $\left\{ \begin{array}{c} \textit{upper bound} \\ \textit{lower bound} \end{array} \right\}$ *of S in $\mathbb{R}$.*

Then we say that σ is a(n) $\left\{ \begin{array}{c} \textbf{supremum} \\ \textbf{infimum} \end{array} \right\}$ **of S in $\mathbb{R}$** *if σ is the* $\left\{ \begin{array}{c} \textit{least element} \\ \textit{greatest element} \end{array} \right\}$ *of the set of all* $\left\{ \begin{array}{c} \textit{upper bounds} \\ \textit{lower bounds} \end{array} \right\}$ *of S in $\mathbb{R}$.*

Remarks.

(A) If S has a supremum in $\mathbb{R}$, it is the unique supremum of S in $\mathbb{R}$. Et cetera.

(B) **Notation.** We denote the $\left\{ \begin{array}{c} \textbf{supremum} \\ \textbf{infimum} \end{array} \right\}$ of S by $\left\{ \begin{array}{c} \sup(S) \\ \inf(S) \end{array} \right\}$.

(C) **Terminology.** We may choose to write '*S has a supremum*' as $\sup(S)$ *exists*'. Et cetera. The situation is analogous for infimum.

11. You may write down any non-empty subset of $\mathbb{R}$ you like, and will find that if the set concerned is bounded above/below in $\mathbb{R}$, it seems to have a supremum/infimum in $\mathbb{R}$.

This provides evidence for the **Least-upper-bound Axiom**, which is a fundamental property of the real number system.

Least-upper-bound Axiom for the reals (LUBA).

Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

The statement (LUBA) is logically equivalent to the equally 'obvious' statement:

'Greatest-lower-bound Axiom for the reals' (GLBA).

Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded below in $\mathbb{R}$. Then A has an infimum in $\mathbb{R}$.

Remarks.

(a) The statements (LUBA), (GLBA) indeed logically equivalent.

The proof is left as an exercise.

(b) The verifications for the non-trivial claims in the re-consideration of Examples (B), (C), (D) require the application of a heuristically obvious but non-trivial result about the real number system known as:

Archimedean Principle (AP).

For any positive real number ε, there exists some positive integer N such that $N\varepsilon > 1$.

The validity of the Archimedean Principle itself relies on the Least-upper-bound Axiom. (This is why those claims are non-trivial.)

In your *mathematical analysis* course, the Least-upper-bound Axiom serves as the ultimate justification for other 'intuitively obvious' results which you have been using without questioning in *infinitesimal calculus*, such as the **Bounded-Monotone Theorem for infinite sequences of real numbers**, the **Intermediate Value Theorem** and the **Mean-Value Theorem**.