1. **Definition.**

Let S be a subset of $\mathbb{R}$.

(a) *Let $\lambda \in S$.*

λ is said to be a $\left\{ \begin{array}{c} \textbf{greatest} \\ \textbf{least} \end{array} \right\}$ **element** *of S if, for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array} \right\}$.

(b) *S is said to have a* $\left\{ \begin{array}{c} \textit{greatest} \\ \textit{least} \end{array} \right\}$ *element if there exists some $\lambda \in S$ such that for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array} \right\}$.

Remarks.

(A) 'If it exists then it is unique': Suppose λ, λ' are $\left\{ \begin{array}{c} \text{greatest} \\ \text{least} \end{array} \right\}$ elements of S. Then $\lambda = \lambda'$.

(B) **Notation.** We denote the $\left\{ \begin{array}{c} \text{greatest} \\ \text{least} \end{array} \right\}$ element of S by $\left\{ \begin{array}{c} \max(S) \\ \min(S) \end{array} \right\}$.

(C) **Terminology.** We may choose to write

'*S has a greatest element*'

as

$\max(S)$ *exists*'.

Et cetera. The situation is analogous for least element.

2. **Definition.**

Let S be a subset of $\mathbb{R}$.

(a) *Let $\beta \in \mathbb{R}$.*

β is said to be a(n) $\left\{ \begin{array}{c} \textbf{upper} \\ \textbf{lower} \end{array} \right\}$ **bound of S in $\mathbb{R}$** *if, for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \beta \\ x \geq \beta \end{array} \right\}$.

(b) *S is said to be* **bounded** $\left\{ \begin{array}{c} \textbf{above} \\ \textbf{below} \end{array} \right\}$ **in $\mathbb{R}$** *if there exists some $\beta \in \mathbb{R}$ such that for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \beta \\ x \geq \beta \end{array} \right\}$.

(c) S is said to be **bounded in $\mathbb{R}$** if S is bounded above in $\mathbb{R}$ and bounded below in $\mathbb{R}$.

Remarks.

(A) If S has one upper bound then it has infinitely many upper bounds.
It does not make sense to write '*the* upper bound of S is so-and-so'.
The situation is similar for 'being bounded below'.

(B) *Suppose λ is a greatest element of S. Then λ is an upper bound of S.*
(How about its converse?)
The situation is similar for 'least element' and 'lower bound'.

3. **Example (A). (Well-ordering Principle for Integers.)**

Recall this statement below, known as the Well-ordering Principle for Integers (WOPI):

Let S be a non-empty subset of $\mathbb{N}$. S has a least element.

There are various re-formulations of the statement (WOPI):

- (WOPIL) *Let T be a non-empty subset of $\mathbb{Z}$.*
  Suppose T is bounded below in $\mathbb{R}$ by some $\beta \in \mathbb{Z}$.
  Then T has a least element.
- (WOPIG) *Let U be a non-empty subset of $\mathbb{Z}$.*
  Suppose U is bounded above in $\mathbb{R}$ by some $\gamma \in \mathbb{Z}$.
  Then U has a greatest element.

(The proof for the logical equivalence of (WOPI), (WOPIL), (WOPIG) is left as an exercise.)

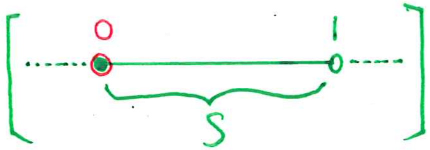
From now on all three of them are referred to as the Well-ordering Principle for Integers.

4. **Example (B).**

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?
$[0,1)$	0 Aa	nil Ab	Ac	Yes (by 1) Ad
$[0,+\infty)$	Ba	nil Bb	Bc	No Bd
$(0,+\infty)$	Ca	Cb	Cc	Cd
$[0,1)\cap\mathbb{Q}$	Da	Db	Dc	Dd
$[0,+\infty)\cap\mathbb{Q}$	Ea	Eb	Ec	Ed
$[0,1)\setminus\mathbb{Q}$	Fa	Fb	Fc	Fd
$[0,+\infty)\setminus\mathbb{Q}$	Ga	Gb	Gc	Gd

(Aa): $[0,1)$ *has a least element, namely,* 0.

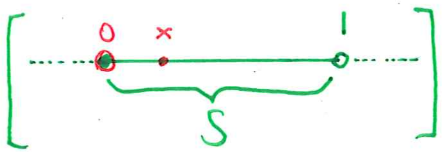
Proof. Write $S = [0,1)$.



- Note that $0 \in S$.

[Check: For any $x \in S$, $x \geq 0$.]

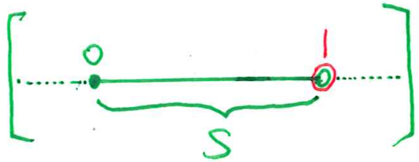
- Pick any $x \in S$.
  By definition of S, $0 \leq x < 1$.
  In particular, $x \geq 0$.



It follows that 0 is a least element of S. □

(Ad): $[0,1)$ *is bounded above in* $\mathbb{R}$ *by* 1.

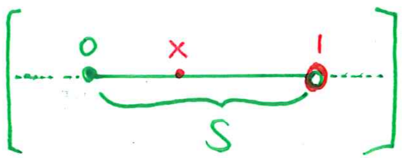
Proof. Write $S = [0,1)$.



- Note that $1 \in \mathbb{R}$.

[Check: For any $x \in S$, $x \leq 1$.]

- Pick any $x \in S$.
  By definition of S, $0 \leq x < 1$.
  In particular, $x < 1$.
  Then $x \leq 1$.



It follows that S is bounded above in $\mathbb{R}$ by 1. □

(Ab): $[0,1)$ *has no greatest element.*

Proof. [Proof-by-contradiction argument.]

Write $S=[0,1)$.

Suppose S had a greatest element, say, λ.

Define $x_0=\dfrac{\lambda+1}{2}$.

Then $0\leq\lambda<x_0<1$.

We have $x_0\in S$ and $x_0>\lambda$.

Contradiction arises. □

(Bb): $[0,+\infty)$ *has no greatest element.*

Proof. [Proof-by-contradiction argument.]

Write $S=[0,+\infty)$.

Suppose S had a greatest element, say λ.

Since $\lambda\in S$, we have $\lambda\geq 0$.

Define $x_0=\lambda+1$.

Then $0\leq\lambda<x_0$.

We have $x_0\in S$ and $x_0>\lambda$.

Contradiction arises. □

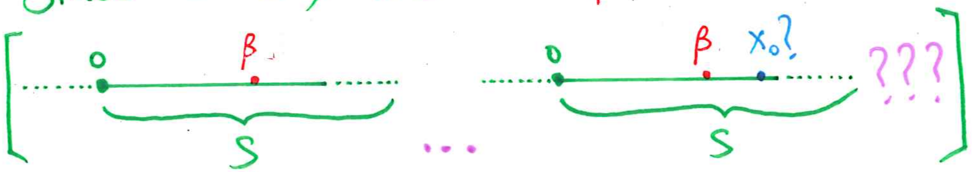
(Bd): $[0,+\infty)$ *is not bounded above in* $\mathbb{R}$.

Proof. [Proof-by-contradiction argument.]

Write $S=[0,+\infty)$.

Suppose S were bounded above in $\mathbb{R}$, say, by β.

Since $0\in S$, we have $\beta\geq 0$.



Define $x_0=\beta+1$.

Then $0\leq\beta<x_0$.

We have $x_0\in S$ and $x_0>\beta$.

Contradiction arises. □

Example (B).

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?
$[0,1)$	0 Aa	nil Ab	↑ Ac	Yes (by 1) Ad
$[0,+\infty)$	0 Ba	nil Bb	All Bc	No Bd
$(0,+\infty)$	nil Ca	nil Cb	yes Cc	No Cd
$[0,1)\cap\mathbb{Q}$	0 Da	nil Db	(by Dc	Yes (by 1) Dd
$[0,+\infty)\cap\mathbb{Q}$	0 Ea	nil Eb	0) Ec	No Ed
$[0,1)\setminus\mathbb{Q}$	nil Fa	nil Fb	Fc	Yes (by 1) Fd
$[0,+\infty)\setminus\mathbb{Q}$	nil Ga	nil Gb	↓ Gc	No Gd

5. **Example (C).**

Let

$$S = \{x \in \mathbb{R} : x^2 \leq (\sqrt{2}+1)x - \sqrt{2}\},$$

and $T = S \backslash \mathbb{Q}$.

(S is in fact the solution set of the inequality

$$x^2 \leq (\sqrt{2}+1)x - \sqrt{2}$$

with unknown x in the reals.)

(a) *S has a greatest element and S has a least element.*

Proof.

- Note that $S = [1, \sqrt{2}]$.
  S has a greatest element, namely $\sqrt{2}$.
  S has a least element, namely 1.

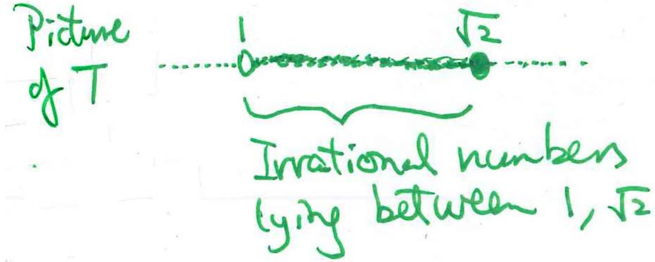
(b) *S is bounded above and below in* $\mathbb{R}$.

Proof.

- S has a least element and a greatest element.
  They are respectively a lower bound and an upper bound of S in $\mathbb{R}$.

$S = [1, \sqrt{2}]$

$T = [1, \sqrt{2}] \setminus \mathbb{Q}$



Example (C).

Let $S = \{x \in \mathbb{R} : x^2 \leq (\sqrt{2}+1)x - \sqrt{2}\}$, *and* $T = S \setminus \mathbb{Q}$.

(c) *T has a greatest element, and T has no least element.*

Proof.

- Note that $T = [1, \sqrt{2}] \setminus \mathbb{Q}$.
- We have $\sqrt{2} \in [1, \sqrt{2}]$, and $\sqrt{2}$ is irrational. Then $\sqrt{2} \in T$.

  Ask: Is it true that for any $x \in T$, $x \leq \sqrt{2}$?

  Pick any $x \in T$.

  By definition, $1 \leq x \leq \sqrt{2}$ and x is irrational. In particular $x \leq \sqrt{2}$.

  Therefore, by definition, $\sqrt{2}$ is a greatest element of T.
- Suppose T had a least element, say, λ. By definition, λ is irrational and $1 \leq \lambda \leq \sqrt{2}$.

  Since λ is irrational, $\lambda \neq 1$. Then $\lambda > 1$.

  Define $x_0 = \dfrac{1+\lambda}{2}$. By definition, $1 < x_0 < \lambda \leq \sqrt{2}$.

  Moreover x_0 is irrational. (Why? Fill in the detail.)

  Then $x_0 \in T$. But $x_0 < \lambda$ and λ is a least element of T. Contradiction arises.

(d) *T is bounded above and below in $\mathbb{R}$.*

Proof.

- T has a least element. It is a lower bound of T in $\mathbb{R}$.
- By definition, for any $x \in T$, $x \leq \sqrt{2}$. Then $\sqrt{2}$ is an upper bound of T in $\mathbb{R}$.

6. **Example (D).**

Let $S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}$.

(a) *S has a greatest element and S has no least element.*

(b) *S is bounded above and below in $\mathbb{R}$.*

(Heuristically, S '=' $\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots;$
$1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \ldots;$
$\frac{2}{3}, \frac{7}{12}, \frac{8}{15}, \ldots;$
$\frac{1}{2}, \frac{9}{20}, \ldots;$
$\frac{2}{5}, \ldots;$
$\ddots \}$)

Proof of (a).

- We verify that S has a greatest element, namely 2:

* Note that $2 \in S$. (Reason: $2 = \frac{1}{0+1} + \frac{1}{0+1}$, and $0 \in \mathbb{N}$.)

* Pick any $x \in S$.

[Check: For any $x \in S$, $x \leq 2$.]

By the definition of S, there exist some $m, n \in \mathbb{N}$ such that $x = \frac{1}{m+1} + \frac{1}{n+1}$.

Since $m, n \in \mathbb{N}$, we have $m \geq 0$ and $n \geq 0$.

Then $x = \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{0+1} + \frac{1}{0+1} = 2$.

* It follows that 2 is a greatest element of S. □

Example (D).

Let $S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}$.

(a) *S has a greatest element and S has no least element.*

(b) *S is bounded above and below in* $\mathbb{R}$.

[Heuristically, S '=' $\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \cdots;$
$1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \cdots;$
$\frac{2}{3}, \frac{7}{12}, \frac{8}{15}, \cdots;$
$\frac{1}{2}, \frac{9}{20}, \cdots;$
$\frac{2}{5}, \cdots;$
$\ddots \}$]

Proof of (a).

[We apply the proof-by-contradiction method to prove that S has no least element.]

Suppose S had a least element, say, λ.

By definition, $\lambda \in S$.

[Ask: Is there any element of S which is less than λ? How to conceive it?]

By the definition of S, there exist some $m_0, n_0 \in \mathbb{N}$ such that $\lambda = \frac{1}{m_0+1} + \frac{1}{n_0+1}$.

Define $x_0 = \frac{1}{m_0+2} + \frac{1}{n_0+1}$. By the definition of S, $x_0 \in S$. (Why?)

Also, $x_0 = \frac{1}{m_0+2} + \frac{1}{n_0+1} < \frac{1}{m_0+1} + \frac{1}{n_0+1} = \lambda$. (Why?)

Contradiction arises. □

7. **Example (B) re-visited for a special observation.**

Re-consider each subset S of $\mathbb{R}$ studied in Example (B):

(a) If S is bounded below in $\mathbb{R}$, then its lower bounds seem to form the closed interval $(-\infty, 0]$, with greatest element, 0.
We may refer to the number 0 as the greatest amongst all lower bounds of S in $\mathbb{R}$, or simply, a greatest lower bound of S in $\mathbb{R}$.
(This claim is easy to verify.)

S	least element?	bounded below in $\mathbb{R}$?	set of all lower bounds?	greatest lower bound?
$[0, 1)$	0	Yes (by 0)	$(-\infty, 0]$	0
$[0, +\infty)$	0	Yes (by 0)	$(-\infty, 0]$	0
$(0, +\infty)$	*nil*	Yes (by 0)	$(-\infty, 0]$	0
$[0, 1) \cap \mathbb{Q}$	0	Yes (by 0)	$(-\infty, 0]$	0
$[0, +\infty) \cap \mathbb{Q}$	0	Yes (by 0)	$(-\infty, 0]$	0
$[0, 1) \backslash \mathbb{Q}$	*nil*	Yes (by 0)	$(-\infty, 0]$	0
$[0, +\infty) \backslash \mathbb{Q}$	*nil*	Yes (by 0)	$(-\infty, 0]$	0

Example (B) re-visited for a special observation.

Re-consider each subset S of $\mathbb{R}$ studied in Example (B):

(a) ...

(b) If S is bounded above in $\mathbb{R}$, then its upper bounds seem to form a closed interval of the form $[1, +\infty)$, with least element 1.

We may refer to the number 1 as the least amongst all upper bounds of S in $\mathbb{R}$, or simply, a least upper bound of S in $\mathbb{R}$.

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?	set of all lower bounds?	set of all upper bounds?	greatest lower bound?	least upper bound?
$[0, 1)$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty)$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$(0, +\infty)$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0, 1) \cap \mathbb{Q}$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty) \cap \mathbb{Q}$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0, 1) \setminus \mathbb{Q}$	*nil*	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty) \setminus \mathbb{Q}$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*

Remark. The verification of the claim ($\sharp$) is non-trivial:

($\sharp$) *The set of all upper bounds of the set* $[0, 1) \cap \mathbb{Q}$ *is* $[1, +\infty)$.

The other two claims are easy to verify.

8. **Example (C) re-visited for a special observation.**

Re-consider Example (C), in which

$$S = \{x \in \mathbb{R} : (\sqrt{2}+1)x - \sqrt{2}\} = [1, \sqrt{2}], \quad T = S \backslash \mathbb{Q}.$$

(a) The lower bounds of S form the closed interval $(-\infty, 1]$, with greatest element 1. (This claim is easy to verify.)
We refer to the number 1 as the greatest lower bound of S.

(b) The upper bounds of S form the closed interval $[\sqrt{2}, +\infty)$, with least element $\sqrt{2}$. (This claim is easy to verify.)
We refer to the number $\sqrt{2}$ as the least upper bound of S.

(c) The lower bounds of T form the closed interval of the form $(-\infty, 1]$, with greatest element 1. (This claim is easy to verify.)
We refer to the number 1 as the greatest lower bound of T.

(d) The upper bounds of T form the closed interval of the form $[\sqrt{2}, +\infty)$, with least element $\sqrt{2}$. (The verification of this claim is non-trivial.)
We refer to the number $\sqrt{2}$ as the least upper bound of T.

9. **Example (D) re-visited for a special observation.**

Re-consider Example (D), in which

$$S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}.$$

(a) The lower bounds of S form the closed interval $(-\infty, 0]$, with greatest element 0. (The verification of this claim is non-trivial.)
We refer to the number 0 as the greatest lower bound of S.

(b) The upper bounds of S form the closed interval $[2, +\infty)$, with least element 2. (This claim is easy to verify.)
We refer to the number 2 as the least upper bound of S.

10. The phenomena discovered in the re-considerations of Examples (B), (C), (D) motivate the definitions for the notion of *supremum*, *infimum* below.

Definition.

Let S be a subset of $\mathbb{R}$, and σ be a real number.

Suppose S is $\left\{\begin{array}{c}\textit{bounded above}\\ \textit{bounded below}\end{array}\right\}$ *in $\mathbb{R}$, and σ is a(n)* $\left\{\begin{array}{c}\textit{upper bound}\\ \textit{lower bound}\end{array}\right\}$ *of S in $\mathbb{R}$.*

Then we say that σ is a(n) $\left\{\begin{array}{c}\textbf{supremum}\\ \textbf{infimum}\end{array}\right\}$ **of** S **in** $\mathbb{R}$ *if σ is the* $\left\{\begin{array}{c}\textit{least element}\\ \textit{greatest element}\end{array}\right\}$ *of the set of all* $\left\{\begin{array}{c}\textit{upper bounds}\\ \textit{lower bounds}\end{array}\right\}$ *of S in $\mathbb{R}$.*

Remarks.

(A) If S has a supremum in $\mathbb{R}$, it is the unique supremum of S in $\mathbb{R}$. Et cetera.

(B) **Notation.** We denote the $\left\{\begin{array}{c}\textbf{supremum}\\ \textbf{infimum}\end{array}\right\}$ of S by $\left\{\begin{array}{c}\sup(S)\\ \inf(S)\end{array}\right\}$.

(C) **Terminology.** We may choose to write '*S has a supremum*' as $\sup(S)$ *exists*'. Et cetera. The situation is analogous for infimum.

11. You may write down any non-empty subset of $\mathbb{R}$ you like, and will find that if the set concerned is bounded above/below in $\mathbb{R}$, it seems to have a supremum/infimum in $\mathbb{R}$.

This provides evidence for the **Least-upper-bound Axiom**, which is a fundamental property of the real number system.

Least-upper-bound Axiom for the reals (LUBA).
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$.
Then A has a supremum in $\mathbb{R}$.

The statement (LUBA) is logically equivalent to the equally 'obvious' statement:
'Greatest-lower-bound Axiom for the reals' (GLBA).
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded below in $\mathbb{R}$.
Then A has an infimum in $\mathbb{R}$.

Remarks.

(a) The statements (LUBA), (GLBA) indeed logically equivalent.
The proof is left as an exercise.

(b) The verifications for the non-trivial claims in the re-consideration of Examples (B), (C), (D) require the application of a heuristically obvious but non-trivial result about the real number system known as:

Archimedean Principle (AP).
For any positive real number ε, there exists some positive integer N such that $N\varepsilon > 1$.

The validity of the Archimedean Principle itself relies on the Least-upper-bound Axiom. (This is why those claims are non-trivial.)

In your *mathematical analysis* course, the Least-upper-bound Axiom serves as the ultimate justification for other 'intuitively obvious' results which you have been using without questioning in *infinitesimal calculus*, such as:

- the **Bounded-Monotone Theorem for infinite sequences of real numbers**,
- the **Intermediate-Value Theorem**, and
- the **Mean-Value Theorem**.