

1. **Definitions. (Arithmetic mean, geometric mean and harmonic mean.)**

Let $n \in \mathbb{N} \setminus \{0\}$. Let a_1, a_2, \dots, a_n be n positive real numbers.

(a) The number $\frac{a_1 + a_2 + \dots + a_n}{n}$ is called the **arithmetic mean** of a_1, a_2, \dots, a_n .

(b) The number $\sqrt[n]{a_1 a_2 \dots a_n}$ is called the **geometric mean** of a_1, a_2, \dots, a_n .

(c) The number $\left[\frac{1}{n} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \right]^{-1}$ is called the **harmonic mean** of a_1, a_2, \dots, a_n .

Remark. By definition, the harmonic mean of a_1, a_2, \dots, a_n is the reciprocal of the arithmetic mean of the reciprocals of a_1, a_2, \dots, a_n .

2. **Theorem (1). (Arithmetico-geometrical Inequality.)**

Let $m \in \mathbb{N} \setminus \{0\}$. Let a_1, a_2, \dots, a_m be m positive real numbers.

The inequality $\frac{a_1 + a_2 + \dots + a_m}{m} \geq \sqrt[m]{a_1 a_2 \dots a_m}$ holds. Equality holds iff $a_1 = a_2 = \dots = a_m$.

Remark. So the result says: the arithmetic mean of an arbitrary collection of ‘finitely many’ positive real numbers is no less than the geometric mean of the same collection. There are many proofs for this result: one of them is given below, as a consequence of Lemma(3), Lemma (4), Lemma (5) and Lemma (6) together.

3. **Corollary (2).**

Let $m \in \mathbb{N} \setminus \{0\}$. Let a_1, a_2, \dots, a_m be m positive real numbers.

The inequality

$$\left[\frac{1}{m} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_m} \right) \right]^{-1} \leq \sqrt[m]{a_1 a_2 \dots a_m} \leq \frac{a_1 + a_2 + \dots + a_m}{m}$$

holds. Each equality holds iff $a_1 = a_2 = \dots = a_m$.

Remark. The proof is left as an exercise: it is a simple extension of the arithmetico-geometrical inequality.

4. **Lemma (3). (‘Special case’ of Theorem (1): ‘for two positive numbers’.)**

Suppose u, v are positive real numbers. Then the inequality $\frac{u+v}{2} \geq \sqrt{uv}$ holds. Equality holds iff $u = v$.

Proof. Exercise.

5. **Illustration of the key idea in the proof of Theorem (1).**

(a) We prove the statement (#) below, which is the ‘inequality part’ of the ‘special case’ of Theorem (1) ‘for four positive numbers’:

(#) Suppose a, b, c, d are positive real numbers. Then $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

Proof of the statement (#). Let a, b, c, d be positive real numbers.

$\sqrt{a}, \sqrt{b}, \sqrt{ab}$ are well-defined, and $a = (\sqrt{a})^2, b = (\sqrt{b})^2, \sqrt{ab} = \sqrt{a}\sqrt{b}$.

Then $a+b = (\sqrt{a})^2 + (\sqrt{b})^2 \geq 2\sqrt{a}\sqrt{b} = 2\sqrt{ab}$. Similarly, $c+d \geq 2\sqrt{cd}$.

Therefore, (once again applying the same argument,) we have

$$\frac{a+b+c+d}{4} = \frac{1}{2} \left(\frac{a+b}{2} + \frac{c+d}{2} \right) \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt[4]{ab} \sqrt[4]{cd} = \sqrt[4]{abcd}.$$

(b) With the help of the statement (#), we deduce the statement (b) below, which is the ‘inequality part’ of the ‘special case’ of Theorem (1) ‘for three positive numbers’:

(b) Suppose r, s, t are positive real numbers. Then $\frac{r+s+t}{3} \geq \sqrt[3]{rst}$.

Proof of the statement (b). Let r, s, t be positive real numbers.

Define $u = \frac{r+s+t}{3}$. u is also a positive real number. By (#), we have $\frac{r+s+t+u}{4} \geq \sqrt[4]{rstu}$.

Note that $\frac{r+s+t+u}{4} = \frac{r+s+t+(r+s+t)/3}{4} = \frac{r+s+t}{3} = u$.

Then $u = \frac{r+s+t+u}{4} \geq \sqrt[4]{rstu} = \sqrt[4]{rst} \cdot \sqrt[4]{u}$.

Note that $u > 0$, and $\sqrt[4]{u} > 0$. Then $(\sqrt[4]{u})^3 \geq \sqrt[4]{rst}$.

Therefore $\frac{r+s+t}{3} = u = \left[\sqrt[3]{(\sqrt[4]{u})^3} \right]^4 \geq \left(\sqrt[3]{\sqrt[4]{rst}} \right)^4 = \left(\sqrt[12]{rst} \right)^4 = \sqrt[3]{rst}$.

6. **Lemma (4).** (Many ‘special cases’ of Theorem (1): ‘for 2^n positive numbers’.)

Let $n \in \mathbb{N}$. Suppose a_1, a_2, \dots, a_{2^n} be 2^n positive real numbers.

Then the inequality $\frac{a_1 + a_2 + \dots + a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \dots a_{2^n}}$ holds.

Equality holds iff $a_1 = a_2 = \dots = a_{2^n}$.

Proof of Lemma (4). Denote by $P(n)$ the proposition below:

‘Suppose a_1, a_2, \dots, a_{2^n} are positive real numbers. Then

$$(a_1 a_2 \dots a_{2^n})^{\frac{1}{2^n}} \leq \frac{a_1 + a_2 + \dots + a_{2^n}}{2^n}.$$

Equality holds iff $a_1 = a_2 = \dots = a_{2^n}$.’

- $P(0)$ is a trivially true statement. (Why?)
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then the statement below is true:

‘Suppose c_1, c_2, \dots, c_{2^k} are positive real numbers. Then

$$(c_1 c_2 \dots c_{2^k})^{\frac{1}{2^k}} \leq \frac{c_1 + c_2 + \dots + c_{2^k}}{2^k}.$$

Equality holds iff $c_1 = c_2 = \dots = c_{2^k}$.’

We are going to verify that $P(k+1)$ is true:

Suppose $b_1, b_2, \dots, b_{2^k}, \dots, b_{2^{k+1}}$ are positive real numbers.

Write $d_1 = (b_1 b_2 \dots b_{2^k})^{\frac{1}{2^k}}$ and $d_2 = (b_{2^k+1} b_{2^k+2} \dots b_{2^{k+1}})^{\frac{1}{2^k}}$. Then

$$\begin{aligned} (b_1 b_2 \dots b_{2^{k+1}})^{\frac{1}{2^{k+1}}} &= \left[(b_1 b_2 \dots b_{2^k})^{\frac{1}{2^k}} (b_{2^k+1} b_{2^k+2} \dots b_{2^{k+1}})^{\frac{1}{2^k}} \right]^{\frac{1}{2}} = \sqrt{d_1 d_2} \\ &\leq \frac{d_1 + d_2}{2} \quad (\text{by Lemma (2)}) \\ &= \frac{1}{2} \left[(b_1 b_2 \dots b_{2^k})^{\frac{1}{2^k}} + (b_{2^k+1} b_{2^k+2} \dots b_{2^{k+1}})^{\frac{1}{2^k}} \right] \\ &\leq \frac{1}{2} \left(\frac{b_1 + b_2 + \dots + b_{2^k}}{2^k} + \frac{b_{2^k+1} + b_{2^k+2} + \dots + b_{2^{k+1}}}{2^k} \right) \\ &= \frac{b_1 + b_2 + \dots + b_{2^{k+1}}}{2^{k+1}} \end{aligned}$$

* Suppose $b_1 = b_2 = \dots = b_{2^{k+1}}$. Then

$$(b_1 b_2 \dots b_{2^{k+1}})^{\frac{1}{2^{k+1}}} = b_1 = \frac{b_1 + b_2 + \dots + b_{2^{k+1}}}{2^{k+1}}.$$

* Suppose

$$(b_1 b_2 \dots b_{2^{k+1}})^{\frac{1}{2^{k+1}}} = \frac{b_1 + b_2 + \dots + b_{2^{k+1}}}{2^{k+1}}.$$

Then

$$\left\{ \begin{array}{l} \frac{d_1 + d_2}{2} = \sqrt{d_1 d_2} \\ \frac{b_1 + b_2 + \dots + b_{2^k}}{2^k} = (b_1 b_2 \dots b_{2^k})^{\frac{1}{2^k}} \\ \frac{b_{2^k+1} + b_{2^k+2} + \dots + b_{2^{k+1}}}{2^k} = (b_{2^k+1} b_{2^k+2} \dots b_{2^{k+1}})^{\frac{1}{2^k}} \end{array} \right.$$

By the second and third equalities, we have $b_1 = b_2 = \dots = b_{2^k}$ and $b_{2^k+1} = b_{2^k+2} = \dots = b_{2^{k+1}}$ respectively. Then $d_1 = b_1$ and $d_2 = b_{2^k+1}$.

Since $\frac{d_1 + d_2}{2} = \sqrt{d_1 d_2}$, we have $b_1 = d_1 = d_2 = b_{2^k+1}$. Therefore $b_1 = b_2 = \dots = b_{2^{k+1}}$.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

7. Lemma (5). ('Backward Induction' Lemma.)

Denote by $Q(m)$ the proposition below:

- Suppose a_1, a_2, \dots, a_m are positive real numbers.
Then $(a_1 a_2 \dots a_m)^{\frac{1}{m}} \leq \frac{a_1 + a_2 + \dots + a_m}{m}$.
Equality holds iff $a_1 = a_2 = \dots = a_m$.

Let $p \in \mathbb{N} \setminus \{0\}$. Suppose $Q(p+1)$ is true. Then $Q(p)$ is true.

Proof of Lemma (5). Denote by $Q(m)$ the proposition below:

- Suppose a_1, a_2, \dots, a_m are positive real numbers.
Then $(a_1 a_2 \dots a_m)^{\frac{1}{m}} \leq \frac{a_1 + a_2 + \dots + a_m}{m}$.
Equality holds iff $a_1 = a_2 = \dots = a_m$.

Let $p \in \mathbb{N} \setminus \{0\}$. The statements $Q(p+1)$, $Q(p)$ respectively read:

$Q(p+1)$: Suppose c_1, c_2, \dots, c_{p+1} are positive real numbers. Then $(c_1 c_2 \dots c_{p+1})^{\frac{1}{p+1}} \leq \frac{c_1 + c_2 + \dots + c_{p+1}}{p+1}$.
Equality holds iff $c_1 = c_2 = \dots = c_{p+1}$.

$Q(p)$: Suppose b_1, b_2, \dots, b_p are positive real numbers. Then $(b_1 b_2 \dots b_p)^{\frac{1}{p}} \leq \frac{b_1 + b_2 + \dots + b_p}{p}$. Equality holds iff $b_1 = b_2 = \dots = b_p$.

Suppose the statement $Q(p+1)$ holds. We proceed to deduce that the statement $Q(p)$ holds:

- * Suppose b_1, b_2, \dots, b_p be positive real numbers.

Define $b_{p+1} = \frac{b_1 + b_2 + \dots + b_p}{p}$. By definition, b_{p+1} is a positive real number. Then by $Q(p+1)$, we have

$$\frac{b_1 + b_2 + \dots + b_p + b_{p+1}}{p+1} \geq (b_1 b_2 \dots b_p \cdot b_{p+1})^{\frac{1}{p+1}}.$$

Note that

$$\frac{b_1 + b_2 + \dots + b_p + b_{p+1}}{p+1} = \frac{b_1 + b_2 + \dots + b_p + (b_1 + b_2 + \dots + b_p)/p}{p+1} = \frac{b_1 + b_2 + \dots + b_p}{p} = b_{p+1}.$$

Then

$$b_{p+1} = \frac{b_1 + b_2 + \dots + b_p + b_{p+1}}{p+1} \geq (b_1 b_2 \dots b_p \cdot b_{p+1})^{\frac{1}{p+1}} = (b_1 b_2 \dots b_p)^{\frac{1}{p+1}} \cdot b_{p+1}^{\frac{1}{p+1}}$$

Note that $b_{p+1}^{\frac{1}{p+1}} > 0$. Then

$$b_{p+1}^{\frac{p}{p+1}} = b_{p+1}^{1 - \frac{1}{p+1}} \geq (b_1 b_2 \dots b_p)^{\frac{1}{p+1}}.$$

Therefore

$$\frac{b_1 + b_2 + \dots + b_p}{p} = b_{p+1} = b_{p+1}^{\frac{p}{p+1} \cdot \frac{p+1}{p}} \geq (b_1 b_2 \dots b_p)^{\frac{1}{p+1} \cdot \frac{p+1}{p}} = (b_1 b_2 \dots b_p)^{\frac{1}{p}}$$

- * Suppose $b_1 = b_2 = \dots = b_p$.

Then

$$(b_1 b_2 \dots b_p)^{\frac{1}{p}} = b_1 = \frac{b_1 + b_2 + \dots + b_p}{p}.$$

- * Suppose

$$(b_1 b_2 \dots b_p)^{\frac{1}{p}} = \frac{b_1 + b_2 + \dots + b_p}{p}.$$

Then $(b_1 b_2 \dots b_p)^{\frac{1}{p}} = b_{p+1}$ by the definition of b_{p+1} .

Therefore

$$(b_1 b_2 \dots b_p \cdot b_{p+1})^{\frac{1}{p+1}} = (b_{p+1}^{p+1})^{\frac{1}{p+1}} = b_{p+1} = \frac{p b_{p+1} + b_{p+1}}{p+1} = \frac{b_1 + b_2 + \dots + b_p + b_{p+1}}{p+1}$$

Therefore $b_1 = b_2 = \dots = b_p = b_{p+1}$. In particular, $b_1 = b_2 = \dots = b_p$.

Hence the statement $Q(p)$ holds.

8. Lemma (6).

Suppose k is a integer greater than 1. There exists some $h \in \mathbb{N}$ so that $2^h < k \leq 2^{h+1}$.

Proof of Lemma (6).

Let $k \in \mathbb{N} \setminus \{0, 1\}$. Note that $2^k \geq k$. (Why?)

Define the set $S = \{x \in \mathbb{N} : 2^x \geq k\}$. [Apply the Well-ordering Principle for integers on the set S .]

We have $k \in S$. Then S is a non-empty subset of \mathbb{N} . By the Well-ordering Principle for integers, S has a least element, which we denote by λ .

Since $k \geq 2$, we have $\lambda \geq 1$.

Define $h = \lambda - 1$. By definition, $h \in \mathbb{N}$ and $k \leq 2^{h+1}$.

We verify that $2^h < k$:

- Suppose it were true that $2^h \geq k$. Then $h \in S$. But $h < \lambda$. Contradiction arises. Hence $2^h < k$ in the first place.

The result follows.

9. Proof of Theorem (1).

Denote by $Q(m)$ the proposition below:

- Suppose a_1, a_2, \dots, a_m are positive real numbers. Then $(a_1 a_2 \dots a_m)^{\frac{1}{m}} \leq \frac{a_1 + a_2 + \dots + a_m}{m}$. Equality holds iff $a_1 = a_2 = \dots = a_m$.

$Q(1)$ is (trivially) true.

By Lemma (4), $Q(2^M)$ is true for any $M \in \mathbb{N}$.

Let $k \in \mathbb{N} \setminus \{0, 1\}$. By Lemma (6), there exists some $h \in \mathbb{N}$ such that $2^h < k \leq 2^{h+1}$.

By Lemma (5), since $Q(2^{h+1})$ is true, $Q(2^{h+1} - 1)$ is true as well. Then, repeatedly applying Lemma (5), we deduce that $Q(2^{h+1} - 2)$ is true, $Q(2^{h+1} - 3)$ is true, ..., $Q(k + 1)$ is true, and $Q(k)$ is true.

It follows that $Q(m)$ is true for any $m \in \mathbb{N} \setminus \{0\}$.

10. ‘Backward induction’ method.

The argument above for the Arithmetico-geometrical Inequality is an example of ‘backward induction’.

Recall this convention on notation:

- Suppose $N \in \mathbb{Z}$. Then $\llbracket N, +\infty \rrbracket$ stands for the set $\{x \in \mathbb{Z} : x \geq N\}$.

Theorem (7). (‘Principle of “Backward induction”’.)

Let $Q(n)$ be a predicate with variable n . Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

Suppose that all of (†), (‡), (★) are true:

- (†) The statement $Q(A_0)$ is true.
- (‡) For any $k \in \mathbb{N}$, if the statement $Q(A_k)$ is true then the statement $Q(A_{k+1})$ is true.
- (★) For any $m \in \llbracket A_0, +\infty \rrbracket$, if the statement $Q(m)$ is true then the statement $Q(m - 1)$ is true.

Then the statement $Q(n)$ is true for any $n \in \llbracket A_0, +\infty \rrbracket$.

Theorem (8). (Set-theoretic formulation of ‘Principle of “Backward induction”’.)

Let T be a subset of $\llbracket A_0, +\infty \rrbracket$. Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

Suppose that all of (†), (‡), (★) are true:

- (†) $A_0 \in T$.
- (‡) For any $k \in \mathbb{N}$, if $A_k \in T$ then $A_{k+1} \in T$.
- (★) For any $m \in \llbracket A_0, +\infty \rrbracket$, if $m \in T$ then $m - 1 \in T$.

Then $T = \llbracket A_0, +\infty \rrbracket$.

The proofs of Theorem (7), Theorem (8) are left as exercises. As statements, Theorem (7) and Theorem (8) are logically equivalent. Theorem (7) suggests a scheme in its application; write down the scheme as an exercise. (A concrete example on how the scheme works is illustrated by the argument in Lemma (4) and Lemma (5).)