1. **Lemma (1).**

   Let a, b be real numbers. Suppose $0 < a < 1$ and $0 < b < 1$. Then $(1-a)(1-b) > 1-(a+b)$.

   Proof of Lemma (1).

   Let a, b be real numbers. Suppose $0 < a < 1$ and $0 < b < 1$.

   Note that $(1-a)(1-b) - [1-(a+b)] = ab$.

   Since $a > 0$ and $b > 0$, we have $ab > 0$.

   Then $(1-a)(1-b) - [1-(a+b)] > 0$.

   Therefore $(1-a)(1-b) > 1-(a+b)$.

2. **Theorem (2).**

   Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $x_1, x_2, \cdots, x_n \in (0,1)$. Then $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) > 1-(x_1+x_2+\cdots+x_n)$.

   Remark. What is Theorem (2) saying? It says that each of the (infinitely many) statements below is true:

   - *Suppose $x_1, x_2 \in (0,1)$. Then $(1-x_1)(1-x_2) > 1-(x_1+x_2)$.*
   - *Suppose $x_1, x_2, x_3 \in (0,1)$. Then $(1-x_1)(1-x_2)(1-x_3) > 1-(x_1+x_2+x_3)$.*
   - *Suppose $x_1, x_2, x_3, x_4 \in (0,1)$. Then $(1-x_1)(1-x_2)(1-x_3)(1-x_4) > 1-(x_1+x_2+x_3+x_4)$.*
   - *Suppose $x_1, x_2, x_3, x_4, x_5 \in (0,1)$. Then $(1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-x_5) > 1-(x_1+x_2+x_3+x_4+x_5)$.*
   - ....

   So the statement of Theorem (2) is of the form

   '*for any $n \in \mathbb{N}\backslash\{0,1\}$, $P(n)$ is true*',

   in which $P(n)$ is the predicate

   '*if $x_1, x_2, \cdots, x_n \in (0,1)$ then $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) > 1-(x_1+x_2+\cdots+x_n)$*'.

   We shall apply mathematical induction to prove Theorem (2).

3. **Proof of Theorem (2).**

   For each integer n greater than 1, we denote by $P(n)$ the proposition below:

   '*If $x_1, x_2, \cdots, x_n \in (0,1)$ then $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) > 1-(x_1+x_2+\cdots+x_n)$.*'

   - By Lemma (1), $P(2)$ is true.
   - Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.
     (Therefore, if $c_1, c_2, \cdots, c_k \in (0,1)$, then the inequality $(1-c_1)(1-c_2)\cdot ... \cdot(1-c_k) > 1-(c_1+c_2+\cdots+c_k)$ holds.)
     We verify that $P(k+1)$ is true:

     [*Reminder.* What we are trying to do is to deduce
     '*if $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$ then $(1-b_1)(1-b_2)\cdot ... \cdot(1-b_k)(1-b_{k+1}) > 1-(b_1+b_2+\cdots+b_k+b_{k+1})$*.
     We expect somewhere along the argument we need to use $P(k)$.']

     Suppose $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$.

     [*Ask.* Is it true that the inequality

     $$(1-b_1)(1-b_2)\cdot ... \cdot(1-b_k)(1-b_{k+1}) > 1-(b_1+b_2+\cdots+b_k+b_{k+1})$$

     holds?
     Observe. There is the expression $(1-b_1)(1-b_2)\cdots(1-b_k)$ in the left-hand side of the desired inequality. It seems to link up with the content of $P(k)$.]

By $P(k)$, the inequality $(1-b_1)(1-b_2)\cdot ... \cdot(1-b_k) > 1-(b_1+b_2+\cdots+b_k)$ holds.
Note that $1-b_{k+1} > 0$. Then

$$\begin{aligned} & (1-b_1)(1-b_2)\cdot ... \cdot(1-b_k)(1-b_{k+1}) \\ > & [1-(b_1+b_2+\cdots+b_k)](1-b_{k+1}) \\ = & 1-(b_1+b_2+\cdots+b_k)-b_{k+1}+(b_1+b_2+\cdots b_k)b_{k+1} \\ > & 1-(b_1+b_2+\cdots+b_k+b_{k+1}). \end{aligned}$$

(Note that $(b_1+b_2+\cdots b_k)b_{k+1} > 0$. It is used in the last inequality.)
Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

4. **Theorem (3).**

   Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $t_1, t_2, \cdots, t_n \in (0, +\infty)$. *Then* $(1+t_1)(1+t_2)\cdot ... \cdot(1+t_n) > 1+(t_1+t_2+\cdots+t_n)$.

   Proof of Theorem (3). Exercise in mathematical induction. (Imitate what is done in the proof of Lemma (1) and Theorem (2).)

5. **Theorem (4).**

   Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $x_1, x_2, \cdots, x_n \in (0,1)$. *Then* $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) < \dfrac{1}{1+(x_1+x_2+\cdots+x_n)}$.

6. **Proof of Theorem (4).** [We apply Theorem (3).]

   Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $x_1, x_2, \cdots, x_n \in (0,1)$.

   For each $j = 1, 2, \cdots, n$, we have $1-{x_j}^2 < 1$. Then $0 < 1-x_j < \dfrac{1}{1+x_j}$.

   Therefore $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) < \dfrac{1}{1+x_1}\cdot\dfrac{1}{1+x_2}\cdot ... \cdot\dfrac{1}{1+x_n}$.

   By Theorem (3), we have $(1+x_1)(1+x_2)\cdot ... \cdot(1+x_n) > 1+(x_1+x_2+\cdots+x_n)$.

   Then

$$(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) < \frac{1}{1+x_1}\cdot\frac{1}{1+x_2}\cdot ... \cdot\frac{1}{1+x_n} < \frac{1}{1+(x_1+x_2+\cdots+x_n)}$$

7. Summarizing the results above, we have the result below:

   Theorem (5). (Weierstrass' Product Inequalities.)

   Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $x_1, x_2, \cdots, x_n \in (0, +\infty)$.

   Then $1-(x_1+x_2+\cdots+x_n) < (1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) < \dfrac{1}{1+(x_1+x_2+\cdots+x_n)}$.