1. **Lemma (1).**

   Let a, b be real numbers.

   Suppose $0 < a < 1$ and $0 < b < 1$.

   Then $(1 - a)(1 - b) > 1 - (a + b)$.

   Proof of Lemma (1).

   Let a, b be real numbers. Suppose $0 < a < 1$ and $0 < b < 1$.

   Note that $(1 - a)(1 - b) - [1 - (a + b)] = ab$.

   Since $a > 0$ and $b > 0$, we have $ab > 0$.

   Then $(1 - a)(1 - b) - [1 - (a + b)] > 0$.

   Therefore $(1 - a)(1 - b) > 1 - (a + b)$.

2. **Theorem (2).**

Let $n \in \mathbb{N} \backslash \{0, 1\}$. *Suppose* $x_1, x_2, \cdots, x_n \in (0, 1)$.
Then $(1 - x_1)(1 - x_2) \cdot ... \cdot (1 - x_n) > 1 - (x_1 + x_2 + \cdots + x_n)$.

Remark. What is Theorem (2) saying? It says that each of the (infinitely many) statements below is true:

- *Suppose* $x_1, x_2 \in (0, 1)$. *Then* $(1 - x_1)(1 - x_2) > 1 - (x_1 + x_2)$.
- *Suppose* $x_1, x_2, x_3 \in (0, 1)$. *Then* $(1 - x_1)(1 - x_2)(1 - x_3) > 1 - (x_1 + x_2 + x_3)$.
- *Suppose* $x_1, x_2, x_3, x_4 \in (0, 1)$.
  Then $(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4) > 1 - (x_1 + x_2 + x_3 + x_4)$.
- *Suppose* $x_1, x_2, x_3, x_4, x_5 \in (0, 1)$.
  Then $(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) > 1 - (x_1 + x_2 + x_3 + x_4 + x_5)$.
- ....

So the statement of Theorem (2) is of the form

'*for any* $n \in \mathbb{N} \backslash \{0, 1\}$, $P(n)$ *is true*',

in which $P(n)$ is the predicate

'*if* $x_1, x_2, \cdots, x_n \in (0, 1)$ *then* $(1 - x_1)(1 - x_2) \cdot ... \cdot (1 - x_n) > 1 - (x_1 + x_2 + \cdots + x_n)$'.

We shall apply mathematical induction to prove Theorem (2).

3. **Proof of Theorem (2).**

For each integer n greater than 1, we denote by $P(n)$ the proposition below:

'*If* $x_1, x_2, \cdots, x_n \in (0,1)$ *then* $(1-x_1)(1-x_2)\cdot \ldots \cdot(1-x_n) > 1-(x_1+x_2+\cdots+x_n)$.'

- By Lemma (1), $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.
  (Therefore, if $c_1, c_2, \cdots, c_k \in (0,1)$, then the inequality
  $(1-c_1)(1-c_2)\cdot \ldots \cdot(1-c_k) > 1-(c_1+c_2+\cdots+c_k)$ holds.)
  We verify that $P(k+1)$ is true:

  [*Reminder.* What we are trying to do is to deduce
  '*if* $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$ *then*
  $(1-b_1)(1-b_2)\cdot \ldots \cdot(1-b_k)(1-b_{k+1}) > 1-(b_1+b_2+\cdots+b_k+b_{k+1})$.
  We expect somewhere along the argument we need to use $P(k)$.']

  Suppose $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$.

  [*Ask.* Is it true that the inequality

  $$(1-b_1)(1-b_2)\cdot \ldots \cdot(1-b_k)(1-b_{k+1}) > 1-(b_1+b_2+\cdots+b_k+b_{k+1})$$

  holds?
  Observe. There is the expression $(1-b_1)(1-b_2)\cdots(1-b_k)$ in the left-hand side of the desired inequality. It seems to link up with the content of $P(k)$.]

Proof of Theorem (2).

For each integer n greater than 1, we denote by $P(n)$ the proposition below:

'*If* $x_1, x_2, \cdots, x_n \in (0,1)$ *then* $(1-x_1)(1-x_2)\cdot \ldots \cdot(1-x_n) > 1-(x_1+x_2+\cdots+x_n)$.'

- By Lemma (1), $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.
  (Therefore, if $c_1, c_2, \cdots, c_k \in (0,1)$, then the inequality
  $(1-c_1)(1-c_2)\cdot \ldots \cdot(1-c_k) > 1-(c_1+c_2+\cdots+c_k)$ holds.)
  We verify that $P(k+1)$ is true:

  Suppose $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$.
  By $P(k)$, the inequality $(1-b_1)(1-b_2)\cdot \ldots \cdot(1-b_k) > 1-(b_1+b_2+\cdots+b_k)$ holds.
  Note that $1-b_{k+1} > 0$. Then

$$
\begin{aligned}
& (1-b_1)(1-b_2)\cdot \ldots \cdot(1-b_k)(1-b_{k+1}) \\
> & [1-(b_1+b_2+\cdots+b_k)](1-b_{k+1}) \\
= & 1-(b_1+b_2+\cdots+b_k)-b_{k+1}+(b_1+b_2+\cdots b_k)b_{k+1} \\
> & 1-(b_1+b_2+\cdots+b_k+b_{k+1}).
\end{aligned}
$$

  (Note that $(b_1+b_2+\cdots b_k)b_{k+1} > 0$. It is used in the last inequality.)
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

4. **Theorem (3).**

Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $t_1, t_2, \cdots, t_n \in (0, +\infty)$.

Then $(1+t_1)(1+t_2)\cdot ... \cdot(1+t_n) > 1 + (t_1 + t_2 + \cdots + t_n)$.

Proof of Theorem (3).

Exercise in mathematical induction. (Imitate what is done in the proof of Lemma (1) and Theorem (2).)

5. **Theorem (4).**

Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $x_1, x_2, \cdots, x_n \in (0, 1)$.

Then $(1-x_1)(1-x_2)\cdot ... \cdot(1-x_n) < \dfrac{1}{1 + (x_1 + x_2 + \cdots + x_n)}$.

6. **Proof of Theorem (4).** [We apply Theorem (3).]

Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $x_1, x_2, \cdots, x_n \in (0,1)$.

For each $j = 1, 2, \cdots, n$, we have $1 - {x_j}^2 < 1$. Then $0 < 1 - x_j < \dfrac{1}{1+x_j}$.

Therefore

$$(1-x_1)(1-x_2)\cdot \ldots \cdot (1-x_n) < \frac{1}{1+x_1}\cdot\frac{1}{1+x_2}\cdot \ldots \cdot \frac{1}{1+x_n}.$$

By Theorem (3), we have $(1+x_1)(1+x_2)\cdot \ldots \cdot (1+x_n) > 1 + (x_1 + x_2 + \cdots + x_n)$.

Then

$$(1-x_1)(1-x_2)\cdot \ldots \cdot (1-x_n) < \frac{1}{1+x_1}\cdot\frac{1}{1+x_2}\cdot \ldots \cdot \frac{1}{1+x_n} < \frac{1}{1+(x_1+x_2+\cdots+x_n)}$$

7. Summarizing the results above, we have the result below:

Theorem (5). (Weierstrass' Product Inequalities.)

Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $x_1, x_2, \cdots, x_n \in (0,+\infty)$.

Then

$$1 - (x_1+x_2+\cdots+x_n) < (1-x_1)(1-x_2)\cdot \ldots \cdot (1-x_n) < \frac{1}{1+(x_1+x_2+\cdots+x_n)}.$$