0. Refer to the Handout *Quadratic polynomials.*

1. **Definition. (Absolute extrema for real-valued functions of one real variable.)**

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely. Let p be a point in I.

   (a) *h is said to* **attain absolute maximum at** p **on** I *if for any $x \in I$, the inequality $h(x) \leq h(p)$ holds. The number $h(p)$ is called the* **absolute maximum value of** h **on** I.

   (b) *h is said to* **attain absolute minimum at** p **on** I *if for any $x \in I$, the inequality $h(x) \geq h(p)$ holds. The number $h(p)$ is called the* **absolute minimum value of** h **on** I.

2. **Theorem (1). (Absolute extrema for quadratic functions.)**

   Let $a, b, c \in \mathbb{R}$, with $a \neq 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$. Denote the discriminant of $f(x)$ by Δ_f.

   (a) *Suppose $a > 0$. Then f attains absolute minimum at $-\dfrac{b}{2a}$ on $\mathbb{R}$, with absolute minimum value $-\dfrac{\Delta_f}{4a}$.*

   (b) *Suppose $a < 0$. Then f attains absolute maximum at $-\dfrac{b}{2a}$ on $\mathbb{R}$, with absolute maximum value $-\dfrac{\Delta_f}{4a}$.*

   Proof of (a) in Theorem (1).

   Let $a, b, c \in \mathbb{R}$, with $a > 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$. Denote the discriminant of $f(x)$ by Δ_f

   For any $x \in \mathbb{R}$, we have $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a} \geq -\dfrac{\Delta_f}{4a}$. Also note that $f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.

   Hence f attains absolute minimum at $-\dfrac{b}{2a}$ on $\mathbb{R}$, with absolute minimum value $-\dfrac{\Delta_f}{4a}$.

3. **Theorem (2), as a Corollary to Theorem (1).**

   Let $a, b, c \in \mathbb{R}$. Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

   Then the statements (†), (‡) are logically equivalent:

   (†) $f(x) \geq 0$ *for any* $x \in \mathbb{R}$.

   (‡) $\Delta_f \leq 0$.

   Equality in (‡) holds iff $-\dfrac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

   Remark. This result will play a key role in the proof of the Cauchy-Schwarz Inequality.

   Proof of Theorem (2).

   Let $a, b, c \in \mathbb{R}$. Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$. By Theorem (1), f attains absolute minimum value at $-\dfrac{b}{2a}$, with $f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.

   - [(†) $\Longrightarrow$ (‡)?] Suppose $f(x) \geq 0$ for any $x \in \mathbb{R}$.

     Note that $-\dfrac{b}{2a} \in \mathbb{R}$. Then, by assumption, we have $0 \leq f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.

     Since $a > 0$, we have $-4a < 0$. Then $\Delta_f = -4a \cdot \left(-\dfrac{\Delta_f}{4a}\right) \leq 0$.

   - [(‡) $\Longrightarrow$ (†)?] Suppose $\Delta_f \leq 0$. Then, since $a > 0$, we have $-\dfrac{\Delta_f}{4a} \geq 0$.

     Pick any $x \in \mathbb{R}$. We have $f(x) \geq f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a} \geq 0$.

   $\Delta_f = 0$ iff $f(x) = a\left(x + \dfrac{b}{2a}\right)^2$ as polynomials. This happens iff $-\dfrac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

4. **Theorem (3). (Cauchy-Schwarz Inequality for 'real vectors'.)**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$. Suppose $x_1, x_2, \cdots, x_n$ are not all zero and $y_1, y_2, \cdots, y_n$ are not all zero. Then the statements below hold:

(a) *The inequality* $\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements* $(\star_1)$, $(\star_2)$ *are logically equivalent:*

$(\star_1)$ $\left|\sum_{j=1}^{n} x_j y_j\right| = \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$.

$(\star_2)$ *There exist some* $p, q \in \mathbb{R}\backslash\{0\}$ *such that* $px_1 + qy_1 = 0$, $px_2 + qy_2 = 0$, *...*, *and* $px_n + qy_n = 0$.

Remarks.

(1) In the context of the statement of Theorem (3), if ($x_1 = x_2 = \cdots = x_n = 0$ or $y_1 = y_2 = \cdots = y_n = 0$), then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

(2) We may re-formulate Theorem (3) in the language of *linear algebra*, and in the process cover the trivial cases mentioned above:

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $\mathbf{x}$, $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$ *defined by* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

Then the statements below hold:

(a) $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

(b) *Equality holds iff* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent over* $\mathbb{R}$.

5. **Theorem (4). (Triangle Inequality for 'real vectors'.)**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$. Suppose $x_1, x_2, \cdots, x_n$ are not all zero and $y_1, y_2, \cdots, y_n$ are not all zero. Then the statements below hold:

(a) *The inequality* $\left[\sum_{j=1}^{n} (x_j + y_j)^2\right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements* $(*_1)$, $(*_2)$ *are logically equivalent:*

$(*_1)$ $\left[\sum_{j=1}^{n} (x_j + y_j)^2\right]^{\frac{1}{2}} = \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$.

$(*_2)$ *There exist* $s > 0, t > 0$ *such that* $sx_1 = ty_1$, $sx_2 = ty_2$, *...*, *and* $sx_n = ty_n$.

Remarks.

(1) In the context of the statement of Theorem (4), if ($x_1 = x_2 = \cdots = x_n = 0$ or $y_1 = y_2 = \cdots = y_n = 0$), then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

(2) We may re-formulate Theorem (4) in the language of *linear algebra*, and in the process cover the trivial cases described above:

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $\mathbf{x}$, $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$ *defined by* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. *Then the statements below hold:*

(a) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

(b) *Equality holds iff one of* $\mathbf{x}$, $\mathbf{y}$ *is a non-negative scalar multiple of the other.*

6. **Proof of Theorem (3).**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$. Suppose $x_1, x_2, \cdots, x_n$ are not all zero and $y_1, y_2, \cdots, y_n$ are not all zero.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \sum_{j=1}^{n}(x_j t + y_j)^2$ for any $t \in \mathbb{R}$.

By definition, for any $t \in \mathbb{R}$, we have $F(t) = \sum_{j=1}^{n}(x_j t + y_j)^2 \geq 0$.

[We are going to identify F as a quadratic polynomial function.]

Define $A = \left(\sum_{j=1}^{n} x_j{}^2\right)$, $B = 2\left(\sum_{j=1}^{n} x_j y_j\right)$, $C = \sum_{j=1}^{n} y_j{}^2$, and $\Delta = B^2 - 4AC$.

For any $t \in \mathbb{R}$, we have $F(t) = \left(\sum_{j=1}^{n} x_j{}^2\right) t^2 + 2\left(\sum_{j=1}^{n} x_j y_j\right) t + \sum_{j=1}^{n} y_j{}^2 = At^2 + Bt + C$.

Since at least one of $x_1, x_2, \cdots, x_n$ is non-zero, we have $A > 0$. Then F is a quadratic polynomial function with real coefficients.

Recall that $F(t) \geq 0$ for any $t \in \mathbb{R}$. Then by Theorem (2), $\Delta \leq 0$.

Therefore $\left|\sum_{j=1}^{n} x_j y_j\right| = \frac{B^2}{4} \leq \sqrt{AC} = \left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}$.

(b) i. [$(\star_1) \Longrightarrow (\star_2)$?]

Suppose $\left|\sum_{j=1}^{n} x_j y_j\right| = \left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}$.

Then $\Delta = B^2 - 4AC = 0$.

By Theorem (2), the quadratic polynomial $F(t)$ has a repeated real root. It is $t_0 = -B/2A$.

Then, for the same t_0, we have $F(t_0) = \sum_{j=1}^{n}(x_j t_0 + y_j)^2 = 0$.

Therefore for each $j = 1, 2, \cdots, n$, we have $(x_j t_0 + y_j)^2 = 0$.

Take $p = t_0$, $q = 1$. For each $j = 1, 2, \cdots, n$, we have $px_j + qy_j = x_j t_0 + y_j = 0$.

Note that $p \neq 0$; otherwise it would happen that $y_j = 0$ for each $j = 1, 2, \cdots, n$. (Why?)

ii. [$(\star_2) \Longrightarrow (\star_1)$?]

Suppose there exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that for each $k = 1, 2, \cdots, n$, the equality $px_k + qy_k = 0$ holds.

Define $t_0 = \frac{p}{q}$. Then for each $k = 1, 2, \cdots, n$, we have $x_k t_0 + y_k = 0$. Therefore $F(t_0) = 0$.

Now the quadratic polynomial $F(t)$ has a real root, namely t_0. Then by Theorem (1), $\Delta \geq 0$.

Also recall $\Delta \leq 0$. Then $\Delta = 0$. Hence $\left|\sum_{j=1}^{n} x_j y_j\right| = \left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}$.

7. **Proof of Theorem (4).**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$. Suppose $x_1, x_2, \cdots, x_n$ are not all zero and $y_1, y_2, \cdots, y_n$ are not all zero.

(a)

$$\begin{aligned}
\left[\left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}\right]^2 &= \sum_{j=1}^{n} x_j{}^2 + \sum_{j=1}^{n} y_j{}^2 + 2\left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}} \\
&\geq \sum_{j=1}^{n} x_j{}^2 + \sum_{j=1}^{n} y_j{}^2 + 2\left|\sum_{j=1}^{n} x_j y_j\right| \quad \text{(by Cauchy-Schwarz Inequality)} \\
&\geq \sum_{j=1}^{n} x_j{}^2 + \sum_{j=1}^{n} y_j{}^2 + 2\sum_{j=1}^{n} x_j y_j = \sum_{j=1}^{n}(x_j{}^2 + 2x_j y_j + y_j{}^2) = \sum_{j=1}^{n}(x_j + y_j)^2
\end{aligned}$$

Hence $\left[\sum_{j=1}^{n}(x_j+y_j)^2\right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}$.

(b) i. $[(*_1) \Longrightarrow (*_2)?]$

Suppose $\left[\sum_{j=1}^{n}(x_j+y_j)^2\right]^{\frac{1}{2}} = \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}$.

Then $\left|\sum_{j=1}^{n}x_jy_j\right| = \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}$, and $\left|\sum_{j=1}^{n}x_jy_j\right| = \sum_{j=1}^{n}x_jy_j$.

By Theorem (3), there exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that for each $k = 1, 2, \cdots, n$, the equality $px_k + qy_k = 0$ holds.

Since $\left|\sum_{j=1}^{n}x_jy_j\right| = \sum_{j=1}^{n}x_jy_j$, we have $\frac{|q|}{|p|}\sum_{j=1}^{n}{x_j}^2 = \left|\sum_{j=1}^{n}-\frac{q}{p}{x_j}^2\right| = \sum_{j=1}^{n}-\frac{q}{p}{x_j}^2 = -\frac{q}{p}\sum_{j=1}^{n}{x_j}^2$.

Then p, q are of opposite signs. Without loss of generality, suppose $q < 0$. Then $p > 0$.
Take $s = p$, $t = -q$. Then $s > 0$ and $t > 0$.
Moreover, for each $k = 1, 2, \cdots, n$, we have $sx_k = px_k = -qy_k = ty_k$.

ii. $[(*_2) \Longrightarrow (*_1)?]$

Suppose there exist some $s > 0, t > 0$ such that for each $k = 1, 2, \cdots, n$, the equality $sx_k = ty_k$ holds.

Then for each $k = 1, 2, \cdots, n$, we have $sx_k - ty_k = 0$. Therefore $\left|\sum_{j=1}^{n}x_jy_j\right| = \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}$.

Moreover, $\left|\sum_{j=1}^{n}x_jy_j\right| = \left|\sum_{j=1}^{n}\frac{t}{s}{x_j}^2\right| = \frac{|t|}{|s|}\left|\sum_{j=1}^{n}{x_j}^2\right| = \frac{t}{s}\sum_{j=1}^{n}{x_j}^2 = \sum_{j=1}^{n}\frac{t}{s}{x_j}^2 = \sum_{j=1}^{n}x_jy_j$.

8. With the help of a basic result (Theorem (5)) for definite integrals of continuous real-valued functions on closed and bounded intervals (which you have learnt in your *calculus of one variable* course and will prove in your *analysis* course), we can deduce the Cauchy-Schwarz Inequality and Triangle Inequality for 'continuous real-valued functions on closed and bounded intervals' (Theorem (6), Theorem (7) respectively). The argument is almost exactly the same as that for Theorem (3) and Theorem (4), with 'summation' replaced by 'definite integration'.

Theorem (5).

Let a, b be real numbers, with $a < b$, and $h : [a, b] \longrightarrow \mathbb{R}$ be a function.

Suppose h is continuous on $[a, b]$ and $h(u) \geq 0$ for any $u \in [a, b]$.

Then the inequality $\int_a^b h(u)du \geq 0$ holds.

Moreover, equality holds iff ($h(u) = 0$ for any $u \in [a, b]$).

Remark. There is a nice geometric interpretation of this result. For such a function h, the number $\int_a^b h(u)du$ stands for the area of the region in the coordinate plane bounded by the curve $y = h(x)$ and the lines $y = 0$, $x = a$, $x = b$. For the area of such a region to be 0, it is necessary and sufficient for the curve $y = h(x)$ to be identical to the y-axis between the points $(a, 0)$ and $(b, 0)$.

9. **Theorem (6). (Cauchy-Schwarz Inequality for definite integrals.)**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is constant zero on $[a, b]$.

Suppose f, g are continuous on $[a, b]$. Then the statements below hold:

(a) *The inequality $\left|\int_a^b f(u)g(u)du\right| \leq \left[\int_a^b (f(u))^2du\right]^{\frac{1}{2}}\left[\int_a^b (g(u))^2du\right]^{\frac{1}{2}}$ holds.*

(b) *The statements $(\star_1)$, $(\star_2)$ are logically equivalent:*

($\star_1$) $\left|\int_a^b f(u)g(u)du\right| = \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$.

($\star_2$) *There exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that $pf(u) + qg(u) = 0$ for any $u \in [a, b]$. (The functions f, g are 'linearly dependent over $\mathbb{R}$'.)*

Remark. In the context of the statement of Theorem (6), if one of the functions f, g is constant zero on $[a, b]$, then the inequality in (a) trivially reduces to the equality in ($\star_1$) of (b).

10. **Theorem (7). (Triangle Inequality for definite integrals.)**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is constant zero on $[a, b]$.

Suppose f, g are continuous on $[a, b]$. Then the statements below hold:

(a) *The inequality* $\left[\int_a^b (f(u) + g(u))^2 du\right]^{\frac{1}{2}} \leq \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} + \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$ *holds.*

(b) *The statements ($*_1$), ($*_2$) are logically equivalent:*

($*_1$) $\left[\int_a^b (f(u) + g(u))^2 du\right]^{\frac{1}{2}} = \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} + \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$.

($*_2$) *There exist some $s > 0$, $t > 0$ such that $sf(u) = tg(u)$. (One of the functions f, g is a non-negative scalar multiple of the other.)*

Remark. In the context of the statement of Theorem (7), if one of the functions f, g is constant zero on $[a, b]$, then the inequality in (a) trivially reduces to the equality in ($*_1$) of (b).

Theorem (7) can be deduced from Theorem (6) in the same way as Theorem (4) is deduced from Theorem (3).

11. **Proof of Theorem (6).**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions.

Suppose neither f nor g is identically zero on $[a, b]$.

Suppose f, g are continuous on $[a, b]$.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \int_a^b (tf(u) + g(u))^2 du$ for any $t \in \mathbb{R}$.

Pick any $t \in \mathbb{R}$. For any $x \in [a, b]$, we have $(tf(x) + g(x))^2 \geq 0$.

Then, by Theorem (5), $F(t) = \int_a^b (tf(u) + g(u))^2 du \geq 0$.

Define $A = \int_a^b (f(u))^2 du$, $B = 2\int_a^b f(u)g(u)du$, $C = \int_a^b (g(u))^2 du$, and $\Delta = B^2 - 4AC$.

By definition, for any $t \in \mathbb{R}$, we have

$$F(t) = \int_a^b (tf(u) + g(u))^2 du = \cdots = t^2 \int_a^b (f(u))^2 du + 2t\int_a^b f(u)g(u)du + \int_a^b (g(u))^2 du = At^2 + Bt + C.$$

Since $(f(x))^2 \geq 0$ for any $x \in [a, b]$, and f is not constant zero on $[a, b]$, we have

$$A = \int_a^b (f(u))^2 du > 0$$

by Theorem (5).

Then F is a quadratic polynomial function with real coefficients. The discriminant of F is Δ.

Recall that $F(t) \geq 0$ for any $t \in \mathbb{R}$. Then $\Delta \leq 0$. Therefore $\frac{B^2}{4} \leq AC$.

Hence

$$\left|\int_a^b f(u)g(u)du\right| = \sqrt{\frac{B^2}{4}} \leq \sqrt{AC} = \sqrt{A}\sqrt{C} = \left(\int_a^b (f(u))^2 du\right)^{\frac{1}{2}} \left(\int_a^b (g(u))^2 du\right)^{\frac{1}{2}}.$$

(b) i. $[(\star_1) \Longrightarrow (\star_2)?]$

Suppose $\left|\int_a^b f(u)g(u)du\right| = \left(\int_a^b (f(u))^2 du\right)^{\frac{1}{2}} \left(\int_a^b (g(u))^2 du\right)^{\frac{1}{2}}$.

Then $\Delta = B^2 - 4AC = 0$.

By Theorem (2), the quadratic polynomial $F(t)$ has a repeated real root. It is $t_0 = -B/2A$.

Then, for the same t_0, we have $F(t_0) = \int_a^b (t_0 f(u) + g(u))^2 du = 0$.

Therefore by Theorem (5), we have $t_0 f(x) + g(x) = 0$ for any $x \in [a, b]$.

Note that $t_0 \neq 0$; otherwise it would happen that $g(x) = 0$ for any $x \in [a, b]$.

ii. $[(\star_2) \Longrightarrow (\star_1)?]$

Suppose there exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that for any $x \in [a, b]$, the equality $pf(x) + qg(x) = 0$ holds.

Define $t_0 = \frac{p}{q}$. Then for each $x \in [a, b]$, we have $t_0 f(x) + g(x) = 0$.

Therefore $F(t_0) = \int_a^b (t_0 f(u) + g(u))^2 du = 0$.

Now the quadratic polynomial $F(t)$ has a real root, namely t_0. Then by Theorem (1), $\Delta \geq 0$.

Also recall $\Delta \leq 0$. Then $\Delta = 0$. Hence $\left|\int_a^b f(u)g(u)du\right| = \left(\int_a^b (f(u))^2 du\right)^{\frac{1}{2}} \left(\int_a^b (g(u))^2 du\right)^{\frac{1}{2}}$.

12. **Appendix 1. Cauchy-Schwarz Inequality and Triangle Inequality for 'square-summable infinite sequences of real numbers'.**

With the help of the Bounded-Monotone Theorem and the notion of absolute convergence for infinite series, we can 'extend' the Cauchy-Schwarz Inequality and Triangle Inequality to analogous results for 'square-summable infinite sequences in $\mathbb{R}$'. For detail, refer to the Handout *Cauchy-Schwarz Inequality and Triangle Inequality for square-summable sequences.*

13. **Appendix 2: Further generalizations.**

(a) There are 'complex analogues' for the 'real versions' of Cauchy-Schwarz Inequalities (Theorem (3), Theorem (6)) and Triangle Inequalities (Theorem (4), Theorem (7)) stated here.

(b) The Cauchy-Schwarz Inequality for 'real vectors' can be seen as a special case of Hölder's Inequality for 'real vectors'. The Triangle Inequality for 'real vectors' can be seen as a special case of Minkowski's Inequality for 'real vectors'. You will encounter these inequalities in advanced courses in *mathematical analysis.*