0. Refer to the Handout *Quadratic polynomials.*

1. **Definition. (Absolute extrema for real-valued functions of one real variable.)**

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely. Let p be a point in I.

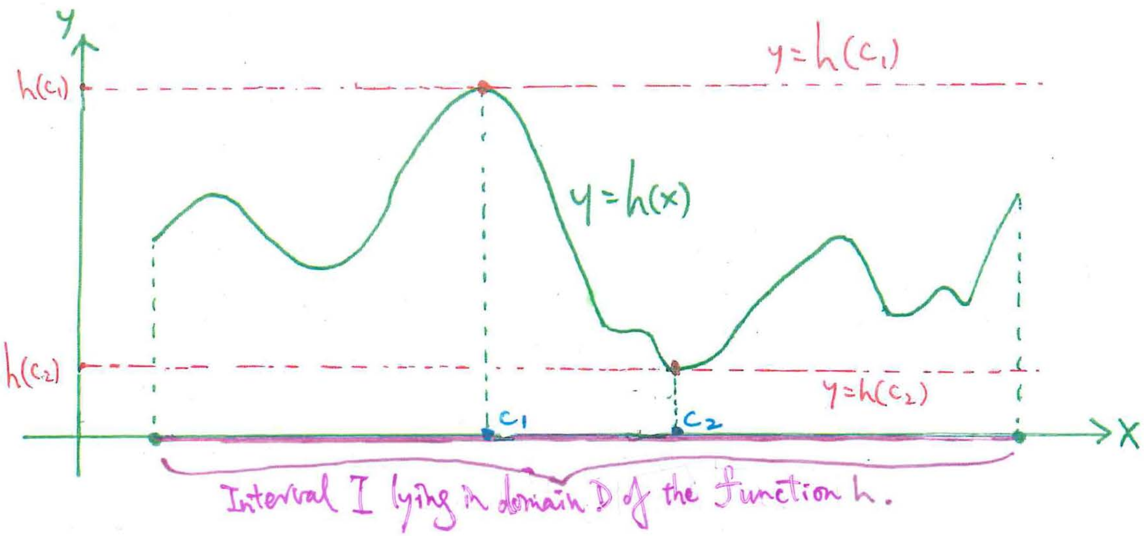
   (a) *h is said to* **attain absolute maximum at p on I** *if for any $x \in I$, the inequality $h(x) \leq h(p)$ holds.*

   The number $h(p)$ is called the **absolute maximum value of h on I**.

   (b) *h is said to* **attain absolute minimum at p on I** *if for any $x \in I$, the inequality $h(x) \geq h(p)$ holds.*

   The number $h(p)$ is called the **absolute minimum value of h on I**.

Picture:



- Absolute maximum attained at c_1 on I, with absolute maximum value $h(c_1)$.
- Absolute minimum attained at c_2 on I, with absolute minimum value $h(c_2)$.

2. **Theorem (1). (Absolute extrema for quadratic functions.)**

Let $a, b, c \in \mathbb{R}$, with $a \neq 0$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

Denote the discriminant of $f(x)$ by Δ_f.

(a) *Suppose $a > 0$. Then f attains absolute minimum at $-\dfrac{b}{2a}$ on $\mathbb{R}$, with absolute minimum value $-\dfrac{\Delta_f}{4a}$.*

(b) *Suppose $a < 0$. Then f attains absolute maximum at $-\dfrac{b}{2a}$ on $\mathbb{R}$, with absolute maximum value $-\dfrac{\Delta_f}{4a}$.*

Proof of (a). Let $a, b, c \in \mathbb{R}$, with $a > 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$. Denote the discriminant of $f(x)$ by Δ_f

For any $x \in \mathbb{R}$, we have $f(x) = ax^2+bx+c = \dots = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta_f}{4a}$.

(a) Suppose $a>0$. Then $f(x) \geq f\left(-\frac{b}{2a}\right) = -\frac{\Delta_f}{4a}$ for any $x \in \mathbb{R}$.

(b) Suppose $a<0$. Then $f(x) \leq f\left(-\frac{b}{2a}\right) = -\frac{\Delta_f}{4a}$ for any $x \in \mathbb{R}$,

The result follows. $\square$

3. **Theorem (2), as a Corollary to Theorem (1).**

Let $a, b, c \in \mathbb{R}$.

Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

Then the statements (†), (‡) are logically equivalent:

(†) $f(x) \geq 0$ *for any* $x \in \mathbb{R}$.

(‡) $\Delta_f \leq 0$.

Equality in (‡) holds iff $-\dfrac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

Remark. This result will play a key role in the proof of the Cauchy-Schwarz Inequality.

Proof of Theorem (2).

Let $a, b, c \in \mathbb{R}$. Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

By Theorem (1), f attains absolute minimum value at $-\dfrac{b}{2a}$, with $f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.

- [(†) $\Longrightarrow$ (‡)?] Suppose $f(x) \geq 0$ for any $x \in \mathbb{R}$.

Then, since $-\frac{b}{2a} \in \mathbb{R}$, we have $0 \leq f(-\frac{b}{2a}) = -\frac{\Delta}{4a}$.
Since $a > 0$, we have $-4a < 0$.
Then $\Delta_f = -4a \cdot (-\frac{\Delta_f}{4a}) \leq 0$.

- [(‡) $\Longrightarrow$ (†)?] Suppose $\Delta_f \leq 0$.

Then, since $a > 0$, we have $-\frac{\Delta_f}{4a} \geq 0$.
Therefore, for any $x \in \mathbb{R}$, we have $f(x) \geq -\frac{\Delta_f}{4a} \geq 0$.

$\Delta_f = 0$ iff $f(x) = a\left(x + \dfrac{b}{2a}\right)^2$ as polynomials.

This happens iff $-\dfrac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

4. **Theorem (3). (Cauchy-Schwarz Inequality for 'real vectors'.)**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $x_1, x_2, \cdots, x_n$ *are not all zero and* $y_1, y_2, \cdots, y_n$ *are not all zero.*

Then the statements below hold:

(a) *The inequality* $\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements* $(\star_1)$, $(\star_2)$ *are logically equivalent:*

$(\star_1)$ $\left|\sum_{j=1}^{n} x_j y_j\right| = \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$.

$(\star_2)$ *There exist some* $p, q \in \mathbb{R}\backslash\{0\}$ *such that* $px_1 + qy_1 = 0$, $px_2 + qy_2 = 0$, ..., *and* $px_n + qy_n = 0$.

Remarks.

(1) In the context of the statement of Theorem (3), if

$$(x_1 = x_2 = \cdots = x_n = 0 \text{ or } y_1 = y_2 = \cdots = y_n = 0),$$

then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

(2) We may re-formulate Theorem (3) in the language of *linear algebra*, and cover the trivial cases mentioned above:

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $\mathbf{x}$, $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$ *defined by* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

Then the statements below hold:

(a) $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

(b) *Equality holds iff* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent over* $\mathbb{R}$.

5. **Theorem (4). (Triangle Inequality for 'real vectors'.)**

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $x_1, x_2, \cdots, x_n$ *are not all zero and* $y_1, y_2, \cdots, y_n$ *are not all zero.*

Then the statements below hold:

(a) *The inequality* $\left[\sum_{j=1}^{n} (x_j + y_j)^2\right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements* $(*_1)$, $(*_2)$ *are logically equivalent:*

$(*_1)$ $\left[\sum_{j=1}^{n} (x_j + y_j)^2\right]^{\frac{1}{2}} = \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$.

$(*_2)$ *There exist* $s > 0, t > 0$ *such that* $sx_1 = ty_1$, $sx_2 = ty_2$, ..., *and* $sx_n = ty_n$.

Remarks.

(1) In the context of the statement of Theorem (4), if

$$(x_1 = x_2 = \cdots = x_n = 0 \text{ or } y_1 = y_2 = \cdots = y_n = 0),$$

then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

(2) We may re-formulate Theorem (4) in the language of *linear algebra*, and cover the trivial cases described above:

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $\mathbf{x}$, $\mathbf{y}$ *are vectors in $\mathbb{R}^n$ defined by* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

Then the statements below hold:

(a) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

(b) *Equality holds iff one of* $\mathbf{x}$, $\mathbf{y}$ *is a non-negative scalar multiple of the other.*

6. **Proof of Theorem (3): 'special case "$n = 2$" ' only.**

Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Suppose x_1, x_2 are not all zero and y_1, y_2 are not all zero.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \underbrace{(x_1 t + y_1)^2}_{\text{'whole square'}} + \underbrace{(x_2 t + y_2)^2}_{\text{'whole square'}}$ for any $t \in \mathbb{R}$.

By definition, for any $t \in \mathbb{R}$, we have $F(t) \geq 0$.

Define $A = x_1^2 + x_2^2$, $B = 2(x_1 y_1 + x_2 y_2)$, $C = y_1^2 + y_2^2$, and $\Delta = B^2 - 4AC$.

For any $t \in \mathbb{R}$, we have

$$F(t) = (x_1 t + y_1)^2 + (x_2 t + y_2)^2 = \dots = (x_1^2 + x_2^2)t^2 + 2(x_1 y_1 + x_2 y_2)t + (y_1^2 + y_2^2)$$
$$= At^2 + Bt + C.$$

Since $x_1 \neq 0$ or $x_2 \neq 0$, we have $A > 0$.

Then F is a quadratic polynomial with positive leading coefficients.

Recall that $F(t) \geq 0$ for any $t \in \mathbb{R}$.

Then, by Theorem (2), $\Delta \leq 0$.

Therefore $|x_1 y_1 + x_2 y_2| = \sqrt{\dfrac{B^2}{4}} \leq \sqrt{AC} = (x_1^2 + x_2^2)^{\frac{1}{2}} (y_1^2 + y_2^2)^{\frac{1}{2}}$.

Proof of Theorem (3): 'special case "$n = 2$" ' only.

Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Suppose x_1, x_2 are not all zero and y_1, y_2 are not all zero.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = (x_1 t + y_1)^2 + (x_2 t + y_2)^2$ for any $t \in \mathbb{R}$. $\cdots$

(b) i. [$(\star_1) \Longrightarrow (\star_2)$?]

Also recall: $A = x_1^2 + x_2^2$, $B = 2(x_1y_1 + x_2y_2)$, $C = y_1^2 + y_2^2$, $\Delta = B^2 - 4AC$.

Suppose $|x_1 y_1 + x_2 y_2| = (x_1{}^2 + x_2{}^2)^{\frac{1}{2}} (y_1{}^2 + y_2{}^2)^{\frac{1}{2}}$.

Then $\Delta = B^2 - 4AC = 0$.

By Theorem (2), the quadratic polynomial $F(t)$ has a repeated real root, namely $-\frac{B}{2A}$.

Write $t_0 = -\frac{B}{2A}$.

Then, for this t_0, we have $0 = F(t_0) = (x_1 t_0 + y_1)^2 + (x_2 t_0 + y_2)^2$.

Therefore $x_1 t_0 + y_1 = 0$ and $x_2 t_0 + y_2 = 0$. (Why?)

Take $p = t_0$, $q = 1$. We have $px_1 + qy_1 = 0$ and $px_2 + qy_2 = 0$.

Note that $p \neq 0$; otherwise we would have $y_1 = 0$ and $y_2 = 0$.

ii. [$(\star_2) \Longrightarrow (\star_1)$?]

Suppose there exist some $p, q \in \mathbb{R} \backslash \{0\}$ such that $px_1 + qy_1 = 0$ and $px_2 + qy_2 = 0$.

Define $t_0 = -p/q$. Then we have $x_1 t_0 + y_1 = 0$ and $x_2 t_0 + y_2 = 0$.

Therefore $F(t_0) = \dots = 0$. The quadratic polynomial $F(t)$ has a real root, namely t_0.

Since F has a real root, $\Delta \geq 0$. But also recall that $\Delta \leq 0$. Then $\Delta = 0$.

Hence $|x_1 y_1 + x_2 y_2| = \sqrt{B^2/4} = \sqrt{AC} = (x_1^2 + x_2^2)^{\frac{1}{2}} (y_1^2 + y_2^2)^{\frac{1}{2}}$. □

7. **Proof of Theorem (4).** Exercise.

8. There is a pair of results about definite integrals which is known as the Cauchy-Schwarz Inequality and the Triangle Inequality.

   They can be proved in a similar way as Theorem (3), Theorem (4) respectively, with the extra help of a result on definite integrals:

   Theorem (5).

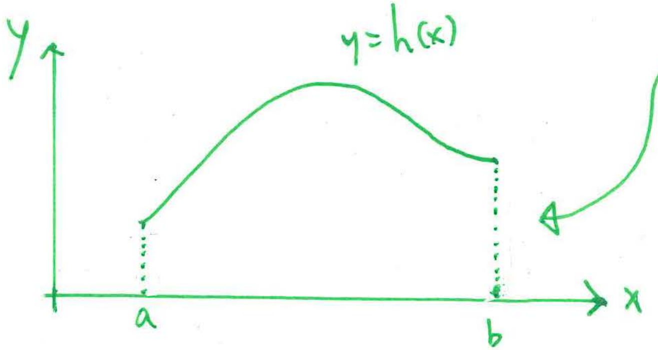
   Let a, b be real numbers, with $a < b$, and $h : [a, b] \longrightarrow \mathbb{R}$ be a function.

   Suppose h is continuous on $[a, b]$ and $h(u) \geq 0$ for any $u \in [a, b]$.

   Then the inequality $\displaystyle\int_a^b h(u)du \geq 0$ holds.

   Moreover, equality holds iff ($h(u) = 0$ for any $u \in [a, b]$).

   Remark. Geometric interpretation?



$\int_a^b h(u)du$ is the area of the region bounded by the curve $y=h(x)$ and the lines $y=0$, $x=a$, $x=b$.
It is 'expected' to be non-negative.

When will it happen that $\int_a^b h(u)du = 0$?
Exactly when '$y=h(x)$' and '$y=0$' are identical from $x=a$ to $x=b$'.

9. **Theorem (6). (Cauchy-Schwarz Inequality for definite integrals.)**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is constant zero on $[a, b]$.

Suppose f, g are continuous on $[a, b]$. Then the statements below hold:

(a) *The inequality* $\left|\int_a^b f(u)g(u)du\right| \leq \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$ *holds.*

(b) *The statements $(\star_1)$, $(\star_2)$ are logically equivalent:*

$(\star_1)$ $\left|\int_a^b f(u)g(u)du\right| = \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$.

$(\star_2)$ *There exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that $pf(u) + qg(u) = 0$ for any $u \in [a, b]$. (The functions f, g are 'linearly dependent over $\mathbb{R}$'.)*

Remark. In the context of the statement of Theorem (6), if one of the functions f, g is constant zero on $[a, b]$, then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

10. **Theorem (7). (Triangle Inequality for definite integrals.)**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is constant zero on $[a, b]$.

Suppose f, g are continuous on $[a, b]$. Then the statements below hold:

(a) *The inequality* $\left[\int_a^b (f(u) + g(u))^2 du\right]^{\frac{1}{2}} \leq \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} + \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$ *holds.*

(b) *The statements $(*_1)$, $(*_2)$ are logically equivalent:*

$(*_1)$ $\left[\int_a^b (f(u) + g(u))^2 du\right]^{\frac{1}{2}} = \left[\int_a^b (f(u))^2 du\right]^{\frac{1}{2}} + \left[\int_a^b (g(u))^2 du\right]^{\frac{1}{2}}$.

$(*_2)$ *There exist some $s > 0$, $t > 0$ such that $sf(u) = tg(u)$. (One of the functions f, g is a non-negative scalar multiple of the other.)*

Remark. In the context of the statement of Theorem (7), if one of the functions f, g is constant zero on $[a, b]$, then the inequality in (a) trivially reduces to the equality in $(*_1)$ of (b).

Theorem (7) can be deduced from Theorem (6) in the same way as Theorem (4) is deduced from Theorem (3).

11. **Proof of Theorem (6).**

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is identically zero on $[a, b]$. Suppose f, g are continuous on $[a, b]$.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \displaystyle\int_a^b (tf(u) + g(u))^2 du$ for any $t \in \mathbb{R}$.

We verify that for any $t \in \mathbb{R}$, $F(t) \geq 0$:

- Pick any $t \in \mathbb{R}$. For any $x \in [a,b]$, we have $(tf(x)+g(x))^2 \geq 0$.

  Then, by Theorem (5),

$$F(t) = \int_a^b (tf(u)+g(u))^2 du \geq 0.$$

Define $A = \int_a^b (f(u))^2 du$, $B = 2\int_a^b f(u)g(u)du$, $C = \int_a^b (g(u))^2 du$, and $\Delta = B^2 - 4AC$.

For any $t \in \mathbb{R}$,

$$F(t) = \int_a^b (tf(u)+g(u))^2 du = \cdots = t^2 \int_a^b (f(u))^2 du + 2t\int_a^b f(u)g(u)du + \int_a^b (g(u))^2 du$$
$$= At^2 + Bt + C.$$

Since $(f(x))^2 \geq 0$ for any $x \in [a,b]$, and f is not constant zero on $[a,b]$, we have

$$A = \int_a^b (f(u))^2 du > 0 \quad \text{(by Theorem (5) again)}.$$

Then F is a quadratic polynomial function with positive leading coefficient.

Recall that $F(t) \geq 0$ for any $t \in \mathbb{R}$. Then by Theorem (2), $\Delta \leq 0$.

Therefore $\left| \int_a^b f(u)g(u)du \right| = \sqrt{B^2/4} \leq \sqrt{AC} = \left(\int_a^b (f(u))^2 du\right)^{1/2} \left(\int_a^b (g(u))^2 du\right)^{1/2}$.

Proof of Theorem (6).

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is identically zero on $[a, b]$. Suppose f, g are continuous on $[a, b]$.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \displaystyle\int_a^b (tf(u) + g(u))^2 du$ for any $t \in \mathbb{R}$.

(b) i. $[(\star_1) \Longrightarrow (\star_2)?]$

Suppose $$\left|\int_a^b f(u)g(u)du\right| = \left(\int_a^b (f(u))^2 du\right)^{\frac{1}{2}} \left(\int_a^b (g(u))^2 du\right)^{\frac{1}{2}}.$$

$\boxed{\text{Also recall: } A = \int_a^b (f(u))^2 du,\ B = 2\int_a^b f(u)g(u)du,\ C = \int_a^b (g(u))^2 du,\ \Delta = B^2 - 4AC.}$

Then $\Delta = B^2 - 4AC = 0$.

By Theorem (2), the quadratic polynomial $F(t)$ has a repeated root, namely $-\frac{B}{2A}$.

Write $t_0 = -\frac{B}{2A}$.

Then for this t_0, we have $0 = F(t_0) = \int_a^b (t_0 f(u) + g(u))^2 du$.

Therefore, by Theorem (5), we have $t_0 f(x) + g(x) = 0$ for any $x \in [a, b]$.

Take $p = t_0$, $q = 1$. We have $p f(x) + q g(x) = 0$ for any $x \in [a, b]$.

Note that $p \neq 0$; otherwise g would be constant zero on $[a, b]$.

Proof of Theorem (6).

Let a, b be real numbers, with $a < b$, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions. Suppose neither f nor g is identically zero on $[a, b]$. Suppose f, g are continuous on $[a, b]$.

(a) Define the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(t) = \displaystyle\int_a^b (tf(u) + g(u))^2 du$ for any $t \in \mathbb{R}$.

(b) i. $[(\star_1) \implies (\star_2)?]$

[Recall: $A = \int_a^b (f(u))^2 du$, $B = 2\int_a^b f(u)g(u)\,du$, $C = \int_a^b (g(u))^2 du$, $\Delta = B^2 - 4AC$.]

ii. $[(\star_2) \implies (\star_1)?]$

Suppose there exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that for any $x \in [a, b]$, the equality $pf(x) + qg(x) = 0$ holds.

Define $t_0 = \frac{p}{q}$.

Then, for any $x \in [a,b]$, $t_0 f(x) + g(x) = 0$.

Therefore $F(t_0) = \int_a^b (t_0 f(u) + g(u))^2 du = 0$.

Now the quadratic polynomial $F(t)$ has a real root, namely t_0.

Since F has a real root, $\Delta \geq 0$.

But also recall $\Delta \leq 0$. Then $\Delta = 0$.

Hence $\left| \int_a^b (f(u))^2 du \right| = \sqrt{B^2/4} = \sqrt{AC} = \left(\int_a^b (f(u))^2 du \right)^{1/2} \left(\int_a^b (g(u))^2 du \right)^{1/2}$.

12. **Appendix 1. Cauchy-Schwarz Inequality and Triangle Inequality for 'square-summable infinite sequences of real numbers'.**

With the help of the Bounded-Monotone Theorem and the notion of absolute convergence for infinite series, we can 'extend' the Cauchy-Schwarz Inequality and Triangle Inequality to analogous results for 'square-summable infinite sequences in $\mathbb{R}$'.

13. **Appendix 2: Further generalizations.**

(a) There are 'complex analogues' for the 'real versions' of Cauchy-Schwarz Inequalities (Theorem (3), Theorem (6)) and Triangle Inequalities (Theorem (4), Theorem (7)) stated here.

(b) The Cauchy-Schwarz Inequality for 'real vectors' can be seen as a special case of Hölder's Inequality for 'real vectors'. The Triangle Inequality for 'real vectors' can be seen as a special case of Minkowski's Inequality for 'real vectors'. You will encounter these inequalities in advanced courses in *mathematical analysis*.