1. Recall the results Theorem ($\star$), Theorem ($\star\star$) from the handout *Arithmetic progression and geometric progression.*

Theorem ($\star$)**.**

Suppose $n \in \mathbb{N}$ *and* $r \in \mathbb{C}$*. Then the statements below hold:*

(a) $1 - r^{n+1} = (1-r)(1 + r + r^2 + \cdots + r^n)$.

(b) *Further suppose* $r \neq 1$*. Then* $\dfrac{1 - r^{n+1}}{1 - r} = 1 + r + r^2 + \cdots + r^n$.

Theorem ($\star\star$)**.**

Suppose $n \in \mathbb{N}$*, and* $s, t \in \mathbb{C}$*.*

Then the equality

$$s^{n+1} - t^{n+1} = (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n)$$

holds.

Assuming the validity of ($\star$), we deduce ($\star\star$) below. We will then apply ($\star\star$) to deduce the result known as 'Bernoulli's Inequality'.

2. **Proof of Theorem** $(\star\star)$.

Suppose $n \in \mathbb{N}$, and $s, t \in \mathbb{C}$.

- (Case 1.) Suppose $s = t = 0$.
  Then $s^{n+1} - t^{n+1} = 0 = (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n)$.
- (Case 2.) Suppose $s \neq 0$.
  Then t/s is well-defined as a number. Write $r = t/s$.
  Then we have

$$\begin{aligned} s^{n+1} - t^{n+1} &= s^{n+1}(1 - r^{n+1}) \\ &= s^{n+1}(1-r)(1 + r + r^2 + \cdots + r^k + \cdots + r^{n-1} + r^n) \\ &= (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n) \end{aligned}$$

- (Case 3.) Suppose $t \neq 0$.
  By modifying the argument in Case 2, (interchanging the respective roles of s, t), we also deduce the equality

$$s^{n+1} - t^{n+1} = (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n).$$

Hence, in any case,

$$s^{n+1} - t^{n+1} = (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n).$$

3. **Theorem (1). (Bernoulli's Inequality.)**

Let $m \in \mathbb{N} \backslash \{0, 1\}$ *and* $\beta \in \mathbb{R}$.

Suppose $\beta > -1$.

Then $(1 + \beta)^m \geq 1 + m\beta$.

Equality holds iff $\beta = 0$.

Proof of Theorem (1).

Let $m \in \mathbb{N} \backslash \{0, 1\}$ and $\beta \in \mathbb{R}$. Suppose $\beta > -1$.

[*Preparatory step.*] Note that

$$\begin{aligned} (1 + \beta)^m - 1 &= (1 + \beta)^m - 1^m \\ &= [(1 + \beta) - 1][(1 + \beta)^{m-1} + (1 + \beta)^{m-2} + \cdots + 1] \\ &= \beta[(1 + \beta)^{m-1} + (1 + \beta)^{m-2} + \cdots + (1 + \beta) + 1] \end{aligned}$$

[*Reminder.* We now deduce the statements below:

(1) *If* $\beta > 0$ *then* $(1 + \beta)^m > 1 + m\beta$.

(2) *If* $-1 < \beta < 0$ *then* $(1 + \beta)^m > 1 + m\beta$.

(3) *If* $\beta = 0$ *then* $(1 + \beta)^m = 1 + m\beta$.

The result follows from a combination of these three statements.]

(1) Suppose $\beta > 0$.

Then, since $\beta > 0$ and $1 + \beta > 1$, we have

$$\begin{aligned}(1+\beta)^m - 1 &= \beta[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + (1+\beta) + 1] \\ &> \beta\,\underbrace{(1 + 1 + \cdots + 1 + 1)}_{m \text{ copies}} = m\beta\end{aligned}$$

Then $(1+\beta)^m > 1 + m\beta$.

(2) Suppose $-1 < \beta < 0$.

Then, since $-\beta > 0$ and $0 < 1 + \beta < 1$, we have

$$\begin{aligned}1 - (1+\beta)^m &= -[(1+\beta)^m - 1] \\ &= (-\beta)[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + (1+\beta) + 1] \\ &< (-\beta)\,\underbrace{(1 + 1 + \cdots + 1 + 1)}_{m \text{ copies}} = -m\beta\end{aligned}$$

Then $(1+\beta)^m > 1 + m\beta$.

(3) Suppose $\beta = 0$. Then $(1+\beta)^m = 1 = 1 + m\beta$.

The result follows.

4. We are going to give some applications of Theorem (1), and various generalizations of Theorem (1).

It deserves notice that when combined with the Sandwich Rule (from your *calculus* course), Theorem (1) will yield some basic results in *mathematical analysis*.

Sandwich Rule.

Let $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ *be infinite sequences in* $\mathbb{R}$.

Suppose that for any $n \in \mathbb{N}$, $u_n \leq v_n \leq w_n$.

Further suppose that $\{u_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ *converge to the same limit, say,* ℓ *in* $\mathbb{R}$.

Then $\{v_n\}_{n=0}^{\infty}$ *also converges to* ℓ.

5. **Theorem (2).**

Let r be a non-zero real number. Suppose $|r| < 1$, and $\alpha = \frac{1}{|r|} - 1$.

Then the statements below hold:

(a) $\alpha > 0$, *and* $|r|^n < \frac{1}{n\alpha}$ *for any* $n \in \mathbb{N} \backslash \{0, 1\}$.

(b) $\lim_{n\to\infty} r^n = 0$.

(c) $\lim_{n\to\infty} \sum_{k=0}^{n} r^k = \frac{1}{1-r}$.

Remark.

Statement (c) immediately yields the result below on geometric progression (as stated in the handout *Arithmetic progression and geometric progression*):

> *Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression of non-zero real numbers, with common ratio r.*
>
> *Suppose $|r| < 1$.*
>
> *Then* $\lim_{n\to\infty}(b_0 + b_1 + b_2 + \cdots + b_n) = \frac{b_0}{1-r}$.

6. **Proof of Theorem (2).**

Let r be a non-zero real number. Suppose $|r| < 1$, and $\alpha = \dfrac{1}{|r|} - 1$.

(a) Note that $\dfrac{1}{|r|} > 1$. Then $\alpha > 0$.

Pick any $n \in \mathbb{N} \backslash \{0, 1\}$.

By Bernoulli's Inequality, we have $(1 + \alpha)^n > 1 + n\alpha$.

Now note that $|r| = \dfrac{1}{1 + \alpha}$.

Then

$$|r|^n = \frac{1}{(1 + \alpha)^n} < \frac{1}{1 + n\alpha} < \frac{1}{n\alpha}.$$

(b) [We apply the Sandwich Rule here.]

For any $n \in \mathbb{N} \backslash \{0, 1\}$, we have

$$-\frac{1}{n\alpha} < -|r|^n \leq r^n \leq |r|^n < \frac{1}{n\alpha}.$$

Note that $\lim_{n \to \infty} -\frac{1}{n\alpha} = 0$ and $\lim_{n \to \infty} -\frac{1}{n\alpha} = 0$.

By the Sandwich Rule, $\lim_{n \to \infty} r^n = 0$ exists and is equal to 0.

(c) For each $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} - \frac{1}{1 - r} \cdot r^n.$$

Then $\lim_{n \to \infty} \sum_{k=0}^{n} r^k$ exists and is equal to $\frac{1}{1 - r}$.

7. **Theorem (3).**

The statements below hold:

(a) *Let* $n \in \mathbb{N} \backslash \{0, 1\}$ *and* $a \in \mathbb{R}$.

Suppose $a > 1$.

Then

$$n(a-1) < a^n - 1 < na^{n-1}(a-1).$$

(b) *Let* $n \in \mathbb{N} \backslash \{0, 1\}$ *and* $b \in \mathbb{R}$.

Suppose $b > 1$.

Then

$$b^n - (b-1)^n < nb^{n-1} < (b+1)^n - b^n.$$

8. **Proof of Theorem (3).**

Construct the arguments as an exercise, using the roughwork below as a framework:

(a) i. For each a and n, we attempt to separately deduce the inequality $a^n - 1 > n(a-1)$ and the inequality $a^n - 1 < na^{n-1}(a-1)$.

ii. An equivalent formulation of the former inequality is $a^n > 1 + n(a-1)$.
This suggests we apply Bernoulli's Inequality appropriately.

iii. An equivalent formulation of the latter inequality is $1 - \dfrac{1}{a^n} < n\left(1 - \dfrac{1}{a}\right)$.
This can be further re-formulated as $\dfrac{1}{a^n} > 1 + n\left(\dfrac{1}{a} - 1\right)$.
This again suggests we apply Bernoulli's Inequality appropriately.

(b) i. For each b and n, we attempt to separately deduce the inequality $b^n - (b-1)^n < nb^{n-1}$ and the inequality $(b+1)^n - b^n > nb^{n-1}$.

ii. An equivalent formulation of the former inequality is $\left(\dfrac{b}{b-1}\right)^n - 1 < n\left(\dfrac{b}{b-1}\right)^{n-1} \cdot \dfrac{1}{b-1}$.
This suggests we apply Statement (a) appropriately.

iii. An equivalent formulation of the latter inequality is $\left(\dfrac{b+1}{b}\right)^n - 1 > n \cdot \dfrac{1}{b}$.
This again suggests we apply Statement (a) appropriately.

9. **Theorem (4).**

(a) *Suppose* $p \in \mathbb{N} \backslash \{0\}$ *and* $m \in \mathbb{N} \backslash \{0, 1\}$.

Then $$\frac{m^{p+1}}{p+1} < \sum_{k=1}^{m} k^p < \frac{(m+1)^{p+1} - 1}{p+1}.$$

(b) *Suppose* $p \in \mathbb{N} \backslash \{0\}$.

Then $$\lim_{m \to \infty} \sum_{k=1}^{m} \frac{1}{m} \left(\frac{k}{m} \right)^p = \frac{1}{p+1}.$$

Proof of Theorem (4). Exercise. Apply Theorem (3) and the Sandwich Rule.

Remark.

Statement (b) is at the core of the argument (from the definition of Riemann integrals) for the result below:

Suppose $x \in \mathbb{R}$ *and* $p \in \mathbb{N} \backslash \{0\}$. *Then* $\displaystyle\int_0^x t^p dt = \frac{x^{p+1}}{p+1}$.

10. **Theorem (5).**

The statements below hold:

(a) *Let* $n \in \mathbb{N} \backslash \{0, 1\}$ *and* $c \in \mathbb{R}$.

Suppose $c > 1$.

Then $\dfrac{c^n - 1}{n} > \dfrac{c^{n-1} - 1}{n - 1}$.

(b) *Let* $k, \ell \in \mathbb{N} \backslash \{0\}$ *and* $d \in \mathbb{R}$.

Suppose $d > 1$, *and* $k > \ell$.

Then $\dfrac{d^k - 1}{k} > \dfrac{d^\ell - 1}{\ell}$.

Proof of Theorem (5). Exercise.

(Apply Theorem (3) to prove Statement (a). Statement (b) is an immediate consequence of Statement (a).)

11. **Theorem (6).**

The statements below hold:

(a) *Let* r *be a rational number, and* $a \in \mathbb{R}$.

Suppose $r > 1$, *and* $a > 1$.

Then $\dfrac{a^r - 1}{r} > a - 1$.

(b) *Let* r *be a rational number, and* $\beta \in \mathbb{R}$.

Suppose $r > 1$, *and* $\beta > 0$.

Then $(1 + \beta)^r > 1 + r\beta$.

Proof of Theorem (6). Exercise.

(Apply Theorem (6) to prove Statement (a). can you name some appropriate positive integers k, ℓ and some real number d for which $a^r = d^k$ and $a^\ell = d$? Statement (b) is an immediate consequence of Statement (a).)

12. Carefully modifying what has been done in Theorem (3), Theorem (5) and Theorem (6), we can obtain the result below, which 'generalizes' Theorem (1) from 'positive integral indices' to 'rational indices'.

Theorem (7). (Generalization of Bernoulli's Inequality to 'rational indices'.)

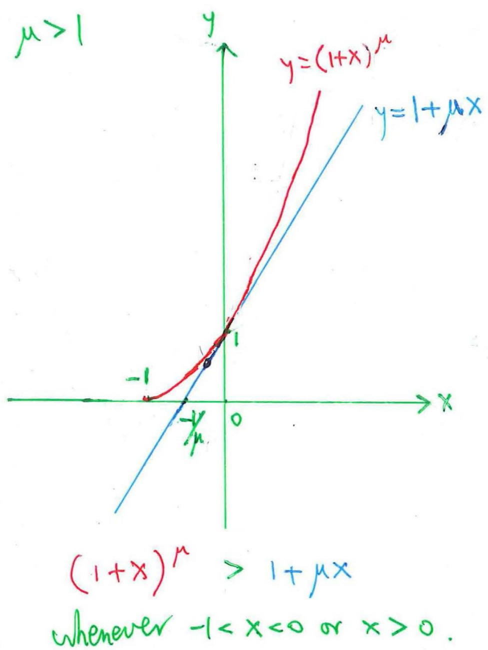
Let μ be a rational number, and β be a real number.

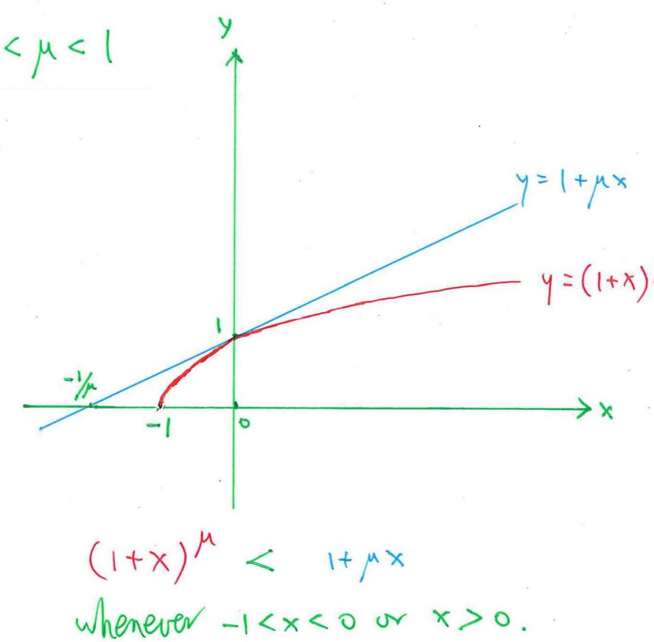
Suppose $\mu \neq 0$ and $\mu \neq 1$, and $\beta > -1$. Then the statements below hold:

(a) *Suppose $\mu < 0$ or $\mu > 1$. Then $(1+\beta)^\mu \geq 1 + \mu\beta$.*

(b) *Suppose $0 < \mu < 1$. Then $(1+\beta)^\mu \leq 1 + \mu\beta$.*

(c) *In each of* (a), (b), *equality holds iff $\beta = 0$.*





As a bonus we can also obtain the result below, which can be regarded as a generalization of Theorem (5):

Theorem (8).

Let s, t be rational numbers and $c \in \mathbb{R}$. Suppose $s > t > 1$. The statements below hold:

(a) *Suppose $c > 1$. Then* $\dfrac{c^s - 1}{s} > \dfrac{c^t - 1}{t}$.

(b) *Suppose $0 < c < 1$. Then* $\dfrac{1 - c^s}{s} > \dfrac{1 - c^t}{t}$.