MATH1050 Absolute Value and Triangle Inequality for the Reals

1. **Definition. (Absolute value of a real number.)**

   Let r be a real number.

   The **absolute value** *of r, which is denoted by $|r|$, is the non-negative real number defined by*

$$|r| = \begin{cases} r & \text{if} \quad r \geq 0 \\ -r & \text{if} \quad r < 0 \end{cases}.$$

   Remarks.

   (a) In a less formal manner we may refer to $|r|$ is the **magnitude** of the real number r.

   (b) This is the geometric interpretation of the definition: $|r|$ is the distance between the point identified as 0 and the point identified as r on the real line.

   Lemma (1).

   Let $r \in \mathbb{R}$. The statements below hold:

   (a) $r \geq 0$ *iff* $|r| = r$.   (b) $r \leq 0$ *iff* $|r| = -r$.   (c) $r = 0$ *iff* $|r| = 0$.   (d) $-|r| \leq r \leq |r|$.

   Proof. Exercise in word game on the definition and the word *iff*.

2. **Lemma (2). (How to 'remove' the absolute value symbol through algebraic means?)**

   Let $r \in \mathbb{R}$. The statements below hold:

   (a) $|r|^2 = r^2$.   (b) $|r| = \sqrt{r^2}$.

   Remark. What is the relevance of this result? We give one example: whenever we obtain in a calculation the expression $|$'blah-blah-blah'$|^2$, we may replace it by the expression ('blah-blah-blah'$)^2$, which may be easier to handle.

   Proof. Let $r \in \mathbb{R}$.

   (a) We have $r \geq 0$ or $r < 0$.

   (Case 1.) Suppose $r \geq 0$. Then $|r| = r$. Therefore $|r|^2 = r^2$.

   (Case 2.) Suppose $r < 0$. Then $|r| = -r$. Therefore $|r|^2 = (-r)^2 = r^2$.

   Hence, in any case, $|r|^2 = r^2$.

   (b) We have verified that $|r|^2 = r^2$. Since $|r| \geq 0$, we have $|r| = \sqrt{|r|^2} = \sqrt{r^2}$.

3. **Lemma (3). (Absolute value and products.)**

   Let $s, t \in \mathbb{R}$. The equality $|st| = |s||t|$ holds.

   Proof. Let $s, t \in \mathbb{R}$. We have $|st|^2 = (st)^2 = s^2t^2 = |s|^2|t|^2 = (|s||t|)^2$. Then $|st| = |s||t|$. (Why?)

4. **Lemma (4). (Basic inequalities concerned with absolute value.)**

   Let $r, c \in \mathbb{R}$. Suppose $c \geq 0$. Then the statements below hold:

   (a) $|r| \leq c$ *iff* $-c \leq r \leq c$.

   (b) $|r| < c$ *iff* $-c < r < c$.

   (c) $|r| \geq c$ *iff* ($r \leq -c$ or $r \geq c$).

   (d) $|r| > c$ *iff* ($r < -c$ or $r > c$).

   Proof. Exercise.

5. **Definition. (Absolute value function.)**

The function from $\mathbb{R}$ *to* $\mathbb{R}$ *defined by assigning each real number to its absolute value is called the* **absolute value function**.

Remark.

In symbols we may denote this function by $|\cdot|$, and express its 'formula of definition' as '$x \longmapsto |x|$ for each $x \in \mathbb{R}$', or equivalently as

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

We may also express the 'formula of definition' of the function $|\cdot|$ as '$|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$'.

6. **Theorem (5). (Triangle Inequality on the real line.)**

Suppose u, v *are real numbers. Then* $|u + v| \leq |u| + |v|$. *Equality holds iff* $uv \geq 0$.

Proof.

Suppose u, v are real numbers.

Then $(|u| + |v|)^2 - |u + v|^2 = (|u| + |v|)^2 - (u + v)^2 = (|u|^2 + |v|^2 + 2|u||v|) - (u^2 + v^2 + 2uv) = 2(|uv| - uv)$. (Why?)

(a) We have $uv \leq |uv|$. Then $(|u| + |v|)^2 - |u + v|^2 \geq 0$.
Therefore $|u + v|^2 \leq (|u| + |v|)^2$.
Since $|u + v| \geq 0$ and $|u| + |v| \geq 0$, we have $|u + v| \leq |u| + |v|$.

(b) i. Suppose $uv \geq 0$. Then $|uv| = uv$. Therefore $(|u| + |v|)^2 - |u + v|^2 = 0$. Hence $|u + v| = |u| + |v|$.
ii. Suppose $|u + v| = |u| + |v|$. Then $(|u| + |v|)^2 - |u + v|^2 = 0$. Therefore $|uv| = uv$. Hence $uv \geq 0$.

Remark.

An alternative argument for this result starts in this way:

> *Suppose* u, v *are real numbers. Then* u, v *are both non-negative, or* u, v *are both non-positive, or (one of* u, v *is non-negative and the other is non-positive).*

Now argue 'case by case'.

7. **Theorem (6). (Triangle Inequality on the real line, also.)**

Suppose s, t *are real numbers. Then* $\big|\ |s| - |t|\ \big| \leq |s - t|$. *Equality holds iff* $st \geq 0$.

Proof. Exercise. (Imitate what the argument for Theorem (5). Play with the expression $(s - t)^2 - (|s| - |t|)^2$.)

Remark. We can also directly apply Theorem (5) to obtain the 'inequality' conclusion in Theorem (6).

8. **Theorem (7). (Generalization of Theorem (5) to the 'many number' situation.)**

Let n *be an integer greater than* 1. *Suppose* $u_1, u_2, \cdots, u_n$ *are real numbers.*

Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.

Equality holds iff $u_1, u_2, \cdots, u_n$ *are all non-negative or all non-positive.*

Proof. Exercise in mathematical induction.

9. **Appendix: Triangle Inequality on the plane.**

Theorem (5) can be regarded as a special case of Theorem (8). (There are 'higher-dimensional analogues' of this result.)

Theorem (8). (Triangle Inequality on the plane.)

Suppose u, v, s, t *are real numbers. Then* $\sqrt{(u + s)^2 + (v + t)^2} \leq \sqrt{u^2 + v^2} + \sqrt{s^2 + t^2}$. *Equality holds iff (*$ut = vs$ *and* $us \geq 0$ *and* $vt \geq 0$*).*

Proof. Postponed.

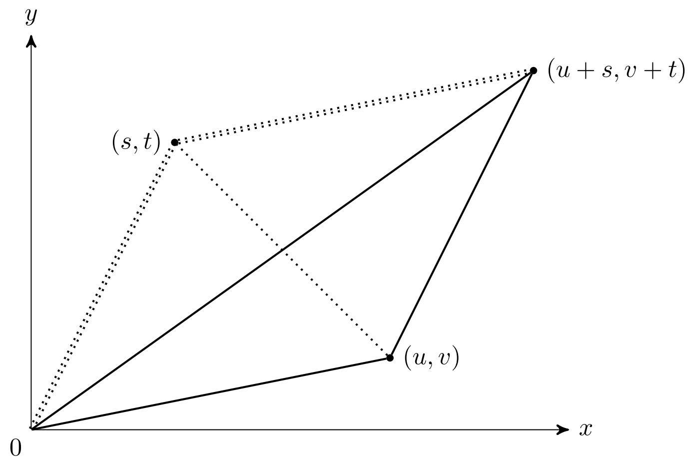
Remark. This is the geometric interpretation of Theorem (8) on the coordinate plane:

Consider the parallelogram whose vertices are $(0,0)$, (u,v), (s,t), $(u+s, v+t)$.

The line segment joining $(0,0)$, (u,v) has length $\sqrt{u^2+v^2}$. The line segment joining (u,v), $(u+s, v+t)$, which is the same as the distance between $(0,0)$, (s,t), has length $\sqrt{s^2+t^2}$. The line segment joining $(0,0)$, $(u+s, v+t)$ is of length $\sqrt{(u+s)^2+(v+t)^2}$.

The sum of the first two lengths is expected to be no shorter than the last. But this is expected: the three line segments are the three sides of the triangle with vertices $(0,0)$, (u,v), $(u+s, v+t)$.

Equality holds exactly when the three points $(u,v), (s,t), (u+s, v+t)$ are on the same 'half-line' with endpoint at the origin $(0,0)$.



10. **Digression: Proofs of statements with conclusion '... iff ...'.**

Re-examine our work on, say, Theorem (5), in this handout. Part of the conclusion in the statement of Theorem (5) reads:

$|u+v| = |u| + |v|$ *iff* $uv \geq 0$.

This is a short-hand for the passage below:

Both statements $(\dagger), (\ddagger)$ *hold:*

$(\dagger)$ *Suppose* $|u+v| = |u| + |v|$. *Then* $uv \geq 0$.

$(\ddagger)$ *Suppose* $uv \geq 0$. *Then* $|u+v| = |u| + |v|$.

For this reason, the argument for this part of Theorem (5) is made up of two 'logically independent' passages, one a justification for $(\dagger)$, and the other a justification for $(\ddagger)$.

This is what we indeed give in the argument for Theorem (5).