1. **Definition. (Absolute value of a real number.)**

Let r be a real number.

The **absolute value** *of r, which is denoted by $|r|$, is the non-negative real number defined by*

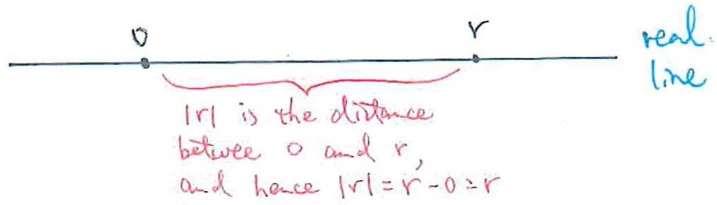
$$|r| = \begin{cases} r & \text{if } r \geq 0 \\ -r & \text{if } r < 0 \end{cases} .$$

Remarks.

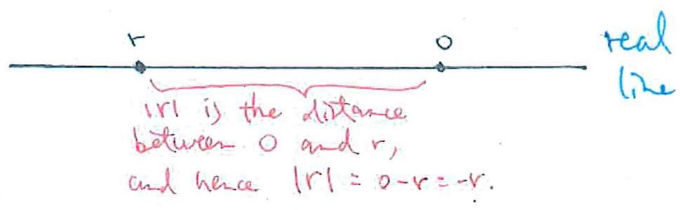
(a) In a less formal manner we may refer to $|r|$ is the **magnitude** of the real number r.

(b) This is the geometric interpretation of the definition: $|r|$ is the distance between the point identified as 0 and the point identified as r on the real line.

(Case 1). Suppose $r \geq 0$. Then:



(Case 2). Suppose $r < 0$. Then:



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Lemma (1).

Let $r \in \mathbb{R}$. The statements below hold:

(a) $r \geq 0$ *iff* $|r| = r$.

(b) $r \leq 0$ *iff* $|r| = -r$.

(c) $r = 0$ *iff* $|r| = 0$.

(d) $-|r| \leq r \leq |r|$.

Proof. Exercise in word game on the definition and the word *iff*.

2. **Lemma (2). (How to 'remove' the absolute value symbol through algebraic means?)**

Let $r \in \mathbb{R}$. The statements below hold:

(a) $|r|^2 = r^2$.

(b) $|r| = \sqrt{r^2}$.

Remark.

What is the relevance of Lemma (2)?

Here is one example:

(a) Whenever we obtain in a calculation

$$|\text{'blah-blah-blah'}|^2,$$

we may replace this expression by

$$(\text{'blah-blah-blah'})^2,$$

which may be easier to handle.

2. **Lemma (2). (How to 'remove' the absolute value symbol through algebraic means?)**

Let $r \in \mathbb{R}$. The statements below hold:

(a) $|r|^2 = r^2$.

(b) $|r| = \sqrt{r^2}$.

Proof.

Let $r \in \mathbb{R}$.

(a) We have $r \geq 0$ or $r < 0$.

(Case 1.) Suppose $r \geq 0$. Then $|r| = r$. Therefore $|r|^2 = r^2$.

(Case 2.) Suppose $r < 0$. Then $|r| = -r$. Therefore $|r|^2 = (-r)^2 = r^2$.

Hence, in any case, $|r|^2 = r^2$.

(b) We have verified that $|r|^2 = r^2$. Since $|r| \geq 0$, we have $|r| = \sqrt{|r|^2} = \sqrt{r^2}$.

3. **Lemma (3). (Absolute value and products.)**

Let $s, t \in \mathbb{R}$. The equality $|st| = |s||t|$ holds.

Proof.

Let $s, t \in \mathbb{R}$. We have $|st|^2 = (st)^2 = s^2t^2 = |s|^2|t|^2 = (|s||t|)^2$. Then $|st| = |s||t|$. (Why?)

4. **Lemma (4). (Basic inequalities concerned with absolute value.)**

Let $r, c \in \mathbb{R}$. Suppose $c \geq 0$. Then the statements below hold:

(a) $|r| \leq c$ iff $-c \leq r \leq c$.

(b) $|r| < c$ iff $-c < r < c$.

(c) $|r| \geq c$ iff ($r \leq -c$ or $r \geq c$).

(d) $|r| > c$ iff ($r < -c$ or $r > c$).

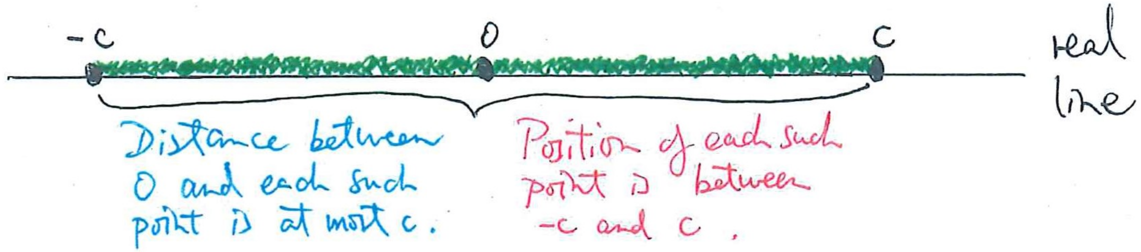
Proof. Exercise.

Geometric interpretation? Below is that for (a); how about others?

(a) The two descriptions in blue, red on r, c are the same:

· 'The distance between 0, r as points in the real line is at most c.'

· 'The position of r in the real line is within the points −c, c.'



5. **Definition. (Absolute value function.)**

The function from $\mathbb{R}$ *to* $\mathbb{R}$ *defined by assigning each real number to its absolute value is called the* **absolute value function**.

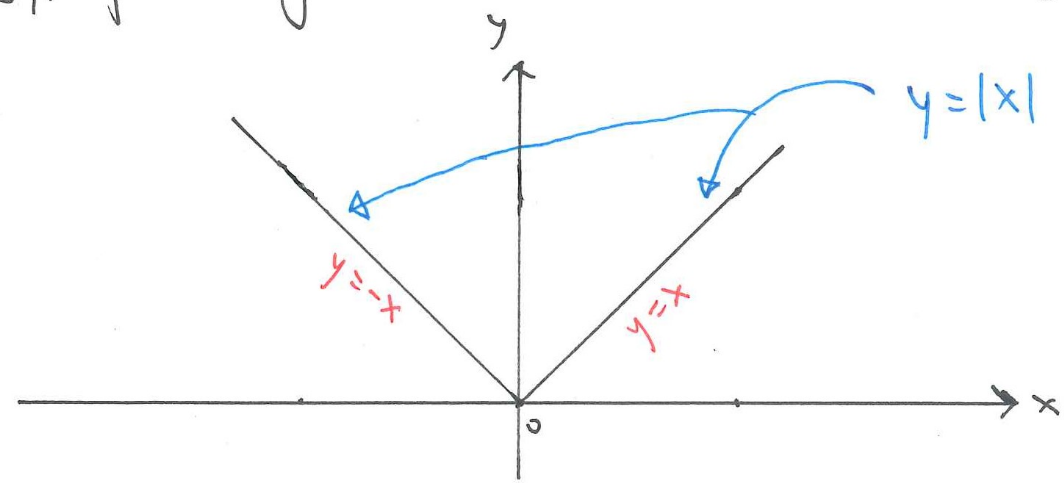
Remark.

In symbols we may denote this function by $|\cdot|$, and express its 'formula of definition' as '$x \longmapsto |x|$ for each $x \in \mathbb{R}$', or equivalently as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We may also express the 'formula of definition' of the function $|\cdot|$ as '$|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$'.

Graph of the absolute value function:



6. **Theorem (5). (Triangle Inequality on the real line.)**
Suppose u, v are real numbers. Then $|u+v| \leq |u| + |v|$. Equality holds iff $uv \geq 0$.

(a) $|u+v| \leq |u| + |v|$; (b) Equality holds iff $uv \geq 0$.

Proof.

Suppose u, v are real numbers.
Then
$$(|u|+|v|)^2 - |u+v|^2$$
$$= (|u|+|v|)^2 - (u+v)^2$$
$$= (|u|^2 + |v|^2 + 2|u||v|) - (u^2 + v^2 + 2uv)$$
$$= 2(|u||v| - uv)$$
$$= 2(|uv| - uv)$$

[Lemma(2) applied.] [Lemma(3) applied.]

(a) We have $uv \leq |uv|$.
Then $(|u|+|v|)^2 \geq |u+v|^2$. (why?)
Since $|u+v| \geq 0$ and $|u|+|v| \geq 0$, we have
$|u+v| \leq |u|+|v|$.

(b)(i). Suppose $uv \geq 0$.
Then $|uv| = uv$ (by definition).
Therefore
$(|u|+|v|)^2 - |u+v|^2 = 2(|uv| - uv) = 0$.
Hence $|u+v| = |u|+|v|$. (why?)

(ii) Suppose $|u+v| = |u|+|v|$.
Then
$0 = (|u|+|v|)^2 - |u+v|^2 = 2(|uv| - uv)$.
Therefore $|uv| = uv$
Hence $uv \geq 0$ (by Lemma(1)).

7. **Theorem (6). (Triangle Inequality on the real line, also.)**
Suppose s, t are real numbers.
Then $\big|\, |s| - |t| \,\big| \leq |s - t|$. Equality holds iff $st \geq 0$.

Proof. Exercise. (Imitate what the argument for Theorem (5). Play with the expression $(s-t)^2 - (|s| - |t|)^2$.)

Remark. We can also directly apply Theorem (5) to obtain the 'inequality' conclusion in Theorem (6).

8. **Theorem (7). (Generalization of Theorem (5) to the 'many number' situation.)**
Let n be an integer greater than 1. Suppose $u_1, u_2, \cdots, u_n$ are real numbers.
Then $|u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$.
Equality holds iff $u_1, u_2, \cdots, u_n$ are all non-negative or all non-positive.

Proof. Exercise in mathematical induction.

9. **Appendix: Triangle Inequality on the plane.**

Theorem (5) can be regarded as a special case of Theorem (7).

Theorem (8). (Triangle Inequality on the plane.)

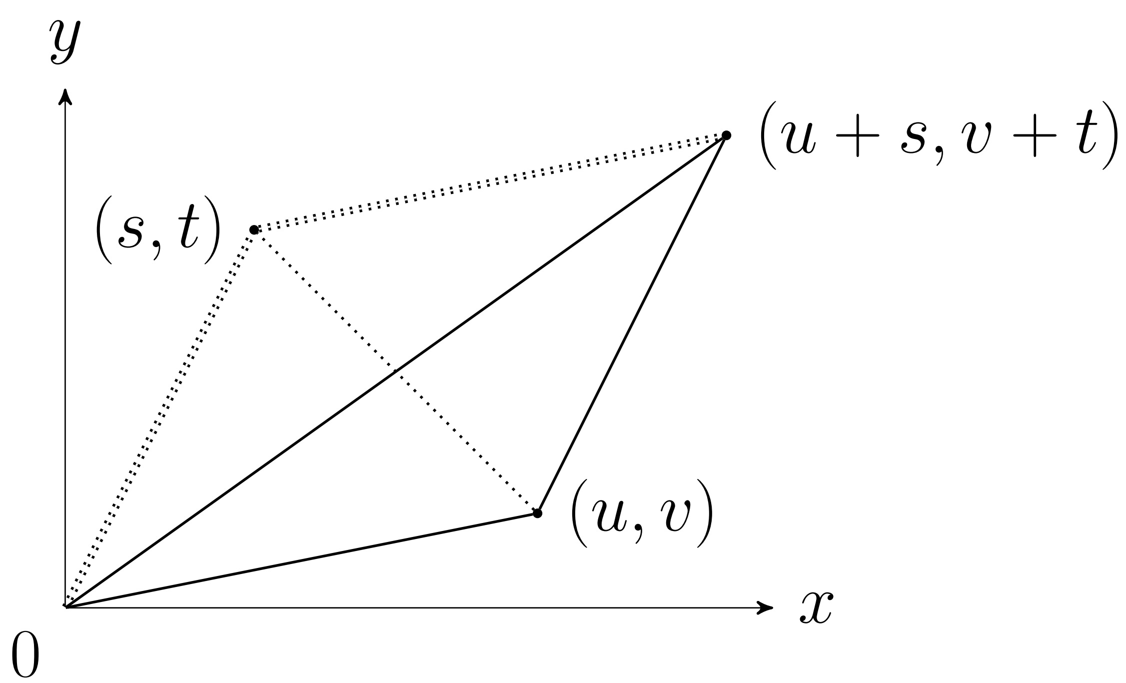
Suppose u, v, s, t are real numbers.

Then $\sqrt{(u+s)^2 + (v+t)^2} \leq \sqrt{u^2 + v^2} + \sqrt{s^2 + t^2}$.

Equality holds iff ($ut = vs$ and $us \geq 0$ and $vt \geq 0$).

Proof. Postponed.

Remark. This is the geometric interpretation of Theorem (8) on the coordinate plane:



10. **Digression: Proofs of statements with conclusion '... iff ...'.**

Re-examine our work on, say, Theorem (5), in this handout.

Part of the conclusion in the statement of Theorem (5) reads:

$|u + v| = |u| + |v|$ *iff* $uv \geq 0$.

This is a short-hand for the passage below:

Both statements $(\dagger), (\ddagger)$ *hold:*

$(\dagger)$ *Suppose* $|u + v| = |u| + |v|$. *Then* $uv \geq 0$.

$(\ddagger)$ *Suppose* $uv \geq 0$. *Then* $|u + v| = |u| + |v|$.

For this reason, the argument for this part of Theorem (5) is made up of two 'logically independent' passages, one a justification for $(\dagger)$, and the other a justification for $(\ddagger)$.

This is what we indeed give in the argument for Theorem (5).