0. In this handout we give some examples on proofs of simple inequalities.

   Along the way, we will pinpoint various statements which look 'basic' (or 'fundamental'), in the sense that we have taken their validity for granted since school days.

1. **Statement (A).**

   Let x be a real number. Suppose $0 < x < 4$. Then $x^2 - 4x < 5$.

   Digression on logic, through Statement (A).

   A chain of words and symbols like 'Statement (A)', for which it makes sense to say it is true or false is called a **mathematical statement**.

   Most statements that you will encounter in this course and beyond can be presented in this form:

   '**Let** *blah-blah-blah.* **Suppose** *bleh-bleh-bleh.* **Then** *blih-blih-blih.*'

   The content within '*blah-blah-blah*, *bleh-bleh-bleh*' collectively is often referred to as the '**assumption part**' of the statement.

   The content within '*blih-blih-blih*' is often referred to as the '**conclusion part**' of the statement.

   For Statement (A), its 'assumption part' is '*x is a real number and* $0 < x < 4$'.

   Its 'conclusion part' is '$x^2 - 4x < 5$'.

Statement (A).

Let x be a real number. Suppose $0 < x < 4$. Then $x^2 - 4x < 5$.

Proof of Statement (A).

Let x be a real number. Suppose $0 < x < 4$.

[*Roughwork.*
We want to deduce, under the assumption '*x is a real number and* $0 < x < 4$', the inequality $x^2 - 4x < 5$.

Question. Is there any equivalent formulation of the conclusion which may be easier to manipulate and which may seem to link with the assumptions?

Answer. $x^2 - 4x - 5 < 0$. (It is good because the 'right-hand side' is 0.)

Observation.
(1) $x^2 - 4x - 5 = (x + 1)(x - 5)$, and
(2) $x + 1 > 1 > 0$, and
(3) $x - 5 < -1 < 0$.
So we know $x^2 - 4x - 5 < 0$. This is exactly what we want.
Now we are ready to organize the ideas appropriately and write up the argument.]

Statement (A).

Let x be a real number. Suppose $0 < x < 4$. Then $x^2 - 4x < 5$.

Proof of Statement (A).

Let x be a real number. Suppose $0 < x < 4$.

Note that $x^2 - 4x - 5 = (x+1)(x-5)$. —— $(\star)$

Since $x > 0$, we have $x + 1 > 1 > 0$.

Since $x < 4$, we have $x - 5 < -1 < 0$.

Then $(x+1)(x-5) < 0$.

Therefore by $(\star)$, we have $x^2 - 4x - 5 < 0$.

Hence $x^2 - 4x < 5$.

2. **Further consideration on the proof of Statement (A).**

We can identify a few ‘basic’ (or ‘fundamental’) statements whose validity we do take for granted and which we have applied in support of our argument for Statement (A):

(a) *Adding the same real number to both sides of a strict inequality results in a strict inequality of the same direction.*

(*For any* $\alpha, \beta, \gamma \in \mathbb{R}$, *if* $\alpha > \beta$ *then* $\alpha + \gamma > \beta + \gamma$.)

(b) *The product of any positive real number and any negative real number is a negative real number.*

(*For any* $\alpha, \beta \in \mathbb{R}$, *if* $\alpha > 0$ *and* $\beta < 0$, *then* $\alpha\beta < 0$.)

3. **Statement (B).**

Suppose x, y are real numbers. Then $4x^2 + 12xy + 11y^2 \geq 0$.

Proof of Statement (B).

Suppose x, y are real numbers.

[*Roughwork.*

We want to deduce, under the assumption '*x, y are real numbers*', the inequality $4x^2 + 12xy + 11y^2 \geq 0$.

Objective. To express $4x^2 + 12xy + 11y^2$ as a sum in which every summand is non-negative.

Ask. But how to do so?

Observation. $4x^2 + 12xy + 11y^2$ is a quadratic expression.

Ask. Can we make use of what we have learnt about completing the square to help us?]

Statement (B).

Suppose x, y are real numbers. Then $4x^2 + 12xy + 11y^2 \geq 0$.

Proof of Statement (B).

Suppose x, y are real numbers.

Note that $4x^2 + 12xy + 11y^2 = (4x^2 + 12xy + 9y^2) + 2y^2 = (2x + 3y)^2 + 2y^2$. —— $(\star)$

Since x, y are real numbers, $2x + 3y$ is a real number. Then $(2x + 3y)^2 \geq 0$.

Since y is a real number, $y^2 \geq 0$. Then $2y^2 \geq 0$.

Since $(2x + 3y)^2 \geq 0$ and $2y^2 \geq 0$, we have $(2x + 3y)^2 + 2y^2 \geq 0$.

Then, by $(\star)$, $4x^2 + 12xy + 11y^2 \geq 0$.

4. **Further consideration on the proof of Statement (B).**

We can identify a few 'basic' (or 'fundamental') statements whose validity we do take for granted and which we have applied in support of our argument for Statement (B):

(a) *The sum of any two real numbers is a real number.*

(*For any* $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta \in \mathbb{R}$.)

(b) *The square of any real number is a non-negative real number.*

(*For any* $\alpha \in \mathbb{R}$, $\alpha^2 \geq 0$.)

(c) *The product of any two non-negative real number is a non-negative real number.*

(*For any* $\alpha, \beta \in \mathbb{R}$, *if* $\alpha \geq 0$ *and* $\beta \geq 0$ *then* $\alpha\beta \geq 0$.)

(d) *The sum of any two non-negative real numbers is a non-negative real number.*

(*For any* $\alpha, \beta \in \mathbb{R}$, *if* $\alpha \geq 0$ *and* $\beta \geq 0$ *then* $\alpha + \beta \geq 0$.)

5. **Statement (C).**

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Proof of Statement (C).

Let x, y be positive real numbers. Suppose $x^2 > y^2$.

[*Roughwork.*
We want to deduce, under the assumption '*x, y are positive real numbers and $x^2 > y^2$*', the inequality $x > y$.

Question. Is there any equivalent formulation of the conclusion which may be easier to manipulate and which may seem to link with the assumptions?

Answer. $x - y > 0$. (It is good because the 'right-hand side' is 0.)

Observation.
(1) $x^2 - y^2 > 0$, and
(2) $x^2 - y^2 = (x - y)(x + y)$, and
(3) $x + y > 0$.
So we expect $x - y > 0$ because both $x^2 - y^2$ and $x + y$ are positive. This is exactly what we want.

Now we are ready to organize the ideas appropriately and write up the argument.]

Statement (C).

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Proof of Statement (C).

Let x, y be positive real numbers. Suppose $x^2 > y^2$.

Then $x^2 - y^2 > 0$.

Note that $x^2 - y^2 = (x - y)(x + y)$.

Then $(x - y)(x + y) > 0$.

Therefore ($x - y > 0$ and $x + y > 0$) or ($x - y < 0$ and $x + y < 0$).

Since $x > 0$ and $y > 0$, we have $x + y > 0$.

Then $x - y > 0$ and $x + y > 0$.

In particular $x - y > 0$.

Therefore $x > y$.

6. **Further consideration on the proof of Statement (C).**

We can identify a few 'basic' (or 'fundamental') statements whose validity we do take for granted and which we have applied in support of our argument for Statement (C):

(a) *The two individual real numbers in a product which is positive are either both positive or both negative.*

(*For any* $\alpha, \beta \in \mathbb{R}$, *if* $\alpha\beta > 0$ *then* ($\alpha > 0$ *and* $\beta > 0$) *or* ($\alpha < 0$ *and* $\beta < 0$).)

(b) *The sum of any two positive real numbers is a positive real number.*

(*For any* $\alpha, \beta \in \mathbb{R}$, *if* $\alpha > 0$ *and* $\beta > 0$ *then* $\alpha + \beta > 0$.)

7. **Statement (D).**

Let x, y be positive real numbers. Suppose $x > y$. Then $x^2 > y^2$.

Proof of Statement (D).

Let x, y be positive real numbers. Suppose $x > y$.

[*Roughwork.*
We want to deduce, under the assumption '*x, y are positive real numbers and $x > y$*', the inequality $x^2 > y^2$.]

Since $x > 0$ and $x > y$, we have $x^2 = x \cdot x > xy$.

Since $y > 0$ and $x > y$, we have $xy > y \cdot y = y^2$.

Then $x^2 > xy$ and $xy > y^2$.

Therefore $x^2 > y^2$.

8. **Further consideration on the proof of Statement (D).**

We can identify a few 'basic' (or 'fundamental') statements whose validity we do take for granted and which we have applied in support of our argument for Statement (D):

(a) *Multiplying the same positive real number to both sides of a strict inequality results in a strict inequality of the same direction.*

(*For any* $\alpha, \beta, \gamma \in \mathbb{R}$, *if* $\alpha > \beta$ *and* $\gamma > 0$ *then* $\alpha\gamma > \beta\gamma$.)

(b) *For three real numbers, if the first is less than the second and the second is less than the third, then the first is less than the third.*

(*For any* $\alpha, \beta, \gamma \in \mathbb{R}$, *if* $\alpha < \beta$ *and* $\beta < \gamma$ *then* $\alpha < \gamma$.)

9. **Digression on logic: assumption, conclusion, converse.**

Compare Statement (C) and Statement (D).

- **Statement (C).**

  Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

  Statement (C) can be re-written as:
  For any positive real numbers x, y, if $x^2 > y^2$ then $x > y$.

- **Statement (D).**

  Let x, y be positive real numbers. Suppose $x > y$. Then $x^2 > y^2$.

  Statement (D) can be re-written as:
  For any positive real numbers x, y, if $x > y$ then $x^2 > y^2$.

By interchanging the respective positions of

$$\text{`}x > y\text{'}, \qquad \text{`}x^2 > y^2\text{'}$$

in Statement (C), we obtain Statement (D). And vice versa.

For this reason, we refer to Statement (C) as the converse of Statement (D), and refer to Statement (D) as the converse of Statement (C).

10. **Statement (E).**

Let x be a real number. Suppose $x^2 - 8x + 7 < 0$. Then $1 < x < 7$.

Proof of Statement (E).

Let x be a real number. Suppose $x^2 - 8x + 7 < 0$.

[*Roughwork.*
We want to deduce, under the assumption '*x is a real number and $x^2 - 8x + 7 < 0$*', the inequalities $1 < x < 7$.]

By assumption, $(x - 1)(x - 7) = x^2 - 8x + 7 < 0$.

Then $(x - 1 < 0$ and $x - 7 > 0)$ or $(x - 1 > 0$ and $x - 7 < 0)$.

[*Reminder.*
We have two possibilities, neither of them being immediately 'eliminated' from what is known.
In this situation, we 'split' the argument into various 'cases', covering various the respective possibilities.]

Statement (E).

Let x be a real number. Suppose $x^2 - 8x + 7 < 0$. Then $1 < x < 7$.

Proof of Statement (E).

Let x be a real number. Suppose $x^2 - 8x + 7 < 0$.

By assumption, $(x-1)(x-7) = x^2 - 8x + 7 < 0$.

Then $(x - 1 < 0$ and $x - 7 > 0)$ or $(x - 1 > 0$ and $x - 7 < 0)$.

(Case 1). Suppose $x - 1 < 0$ and $x - 7 > 0$.
Since $x - 1 < 0$, we have $x < 1$.
Since $x - 7 > 0$, we have $x > 7$.
Then $7 < x < 1$.
Therefore $7 < 1$. This is impossible.

(Case 2). Suppose $x - 1 > 0$ and $x - 7 < 0$.
Since $x - 1 > 0$, we have $x > 1$.
Since $x - 7 < 0$, we have $x < 7$.
Then $1 < x < 7$.

So, to conclude, we have $1 < x < 7$.

11. **Statement (F).**

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$. Then $x^2 + xy + y^2 > 0$.

Proof of Statement (F).

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$.

[*Reminder.*
We have two possibilities. We 'split' the argument into various 'cases', covering various the respective possibilities.
In each case, with the help of the extra assumption in that case, we try to obtain the conclusion '$x^2 + xy + y^2 > 0$'.
The tool to use is 'completing the square'.]

(Case 1). Suppose $x \neq 0$.

Then $x^2 + xy + y^2 = \dfrac{3x^2}{4} + \left(\dfrac{x}{2} + y\right)^2 > 0 + 0 = 0.$

(Case 2). Suppose $y \neq 0$.

Then $x^2 + xy + y^2 = \dfrac{3y^2}{4} + \left(\dfrac{y}{2} + x\right)^2 > 0.$

Hence, in any case, $x^2 + xy + y^2 > 0$.

12. **Statement (G).**

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (G).

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$.
Then $x^2 - y^2 \geq 0$.
Note that $x^2 - y^2 = (x - y)(x + y)$.
Then $(x - y)(x + y) \geq 0$.
Since $x \geq 0$ and $y \geq 0$, we have $x + y \geq 0$.

[*Reminder.* We are not in a position to immediately conclude that $x - y \geq 0$. Why? We cannot 'divide both sides' by $x + y$, because we have not ruled out the possibility that $x + y$ may be 0.]

Then $x + y > 0$ or $x + y = 0$.

(Case 1). Suppose $x + y > 0$.
Since $(x - y)(x + y) \geq 0$, we have $x - y \geq 0$. Therefore $x \geq y$.

(Case 2). Suppose $x + y = 0$.
Since $x \geq 0$ and $y \geq 0$, we have $x = y = 0$. Therefore $x \geq y$.

Hence, in any case, $x \geq y$.

13. **Statement (H). (A 'baby version' of the Cauchy-Schwarz Inequality.)**
Suppose x, y *are real numbers. Then* $x^2 + y^2 \geq 2xy$. *Equality holds iff* $x = y$.

Proof of statement (H).

Suppose x, y are real numbers.

[*Preparatory step.*
Study the difference 'L.H.S. minus R.H.S.' in the desired inequality.]

We have $(x^2 + y^2) - 2xy = (x - y)^2$.

[*Reminder.*
With the help of the preparatory step, we now deduce the statements below:
(1) $x^2 + y^2 \geq 2xy$.
(2) *If* $x = y$ *then* $x^2 + y^2 = 2xy$.
(3) *If* $x^2 + y^2 = 2xy$ *then* $x = y$.
The result follows from a combination of these three statements.]

Statement (H). (A 'baby version' of the Cauchy-Schwarz Inequality.)

Suppose x, y are real numbers. Then $x^2 + y^2 \geq 2xy$. Equality holds iff $x = y$.

Proof of statement (H).

Suppose x, y are real numbers. We have $(x^2 + y^2) - 2xy = (x - y)^2$.

(1) Since x, y are real, $x - y$ is real.
Then $(x - y)^2 \geq 0$.
Therefore $x^2 + y^2 \geq 2xy$.

(2) Suppose $x = y$.
Then $(x^2 + y^2) - 2xy = (x - y)^2 = (x - x)^2 = 0$.
Therefore $x^2 + y^2 = 2xy$.

(3) Suppose $x^2 + y^2 = 2xy$.
Then $0 = (x^2 + y^2) - 2xy = (x - y)^2$.
Therefore $x - y = 0$. Hence $x = y$.

The result follows.

Remark. Strictly speaking, Statement (H) is not just about an inequality.
It is about a non-strict inequality together with the 'necessary and sufficient conditions for the equality to hold'.

This kind of statements is common amongst results concerned with inequalities.

14. **Tacitly assumed statements about inequality in the real number system.**

Carefully examining the proofs of the inequalities above, we probably have to concede that we need expand the list of 'rules as regards inequalities' which we are tacitly assuming since school-days.

To be more efficient, we state them with the help of symbols.

(1) *Suppose* $\alpha, \beta \in \mathbb{R}$. *Then* $\beta - \alpha > 0$ *iff* $\alpha < \beta$.

(1*) *Suppose* $x, y \in \mathbb{R}$. *Then* $\beta - \alpha \geq 0$ *iff* $\alpha \leq \beta$.

(2) *Let* $\alpha, \beta, \gamma \in \mathbb{R}$. *Suppose* $\alpha < \beta$ *and* $\beta < \gamma$. *Then* $\alpha < \gamma$.

(2*) *The statements below hold:*

(2*a) *Suppose* $\alpha \in \mathbb{R}$. *Then* $\alpha \leq \alpha$.

(2*b) *Let* $\alpha, \beta \in \mathbb{R}$. *Suppose* ($\alpha \leq \beta$ *and* $\beta \leq \alpha$). *Then* $\alpha = \beta$.

(2*c) *Let* $\alpha, \beta, \gamma \in \mathbb{R}$. *Suppose* ($\alpha \leq \beta$ *and* $\beta \leq \gamma$). *Then* $\alpha \leq \gamma$.

(3) *Suppose* $\alpha \in \mathbb{R}$. *Then exactly one of* '$\alpha < 0$', '$\alpha = 0$', '$\alpha > 0$' *is true.*

(4) *Let* $\alpha, \beta \in \mathbb{R}$. *Suppose* $\alpha < \beta$. *Then the statements below hold:*

(4a) *For any* $\gamma \in \mathbb{R}$, $\alpha + \gamma < \beta + \gamma$ *and* $\alpha - \gamma < \beta - \gamma$.

(4b) *For any* $\gamma \in \mathbb{R}$, *if* $\gamma > 0$ *then* $\alpha\gamma < \beta\gamma$ *and* $\alpha/\gamma < \beta/\gamma$.

(4c) *For any* $\gamma \in \mathbb{R}$, *if* $\gamma < 0$ *then* $\alpha\gamma > \beta\gamma$ *and* $\alpha/\gamma > \beta/\gamma$.

(4*) *Let* $\alpha, \beta \in \mathbb{R}$. *Suppose* $\alpha \leq \beta$. *Then the statements below hold:*

(4*a) *For any* $\gamma \in \mathbb{R}$, $\alpha + \gamma \leq \beta + \gamma$ *and* $\alpha - \gamma \leq \beta - \gamma$.

(4*b) *For any* $\gamma \in \mathbb{R}$, *if* $\gamma > 0$ *then* $\alpha\gamma \leq \beta\gamma$ *and* $\alpha/\gamma \leq \beta/\gamma$.

(4*c) *For any* $\gamma \in \mathbb{R}$, *if* $\gamma < 0$ *then* $\alpha\gamma \geq \beta\gamma$ *and* $\alpha/\gamma \geq \beta/\gamma$.

(5) *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. *Suppose* $\alpha < \beta$ *and* $\gamma < \delta$. *The statements below hold:*

(5*a*) $\alpha + \gamma < \beta + \delta$.

(5*b*) *Further suppose* $\alpha > 0$, $\beta > 0$, $\gamma > 0$ *and* $\delta > 0$. *Then* $\alpha\gamma < \beta\delta$.

(5*) *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. *Suppose* $\alpha \leq \beta$ *and* $\gamma \leq \delta$. *The statements below hold:*

(5*a) $\alpha + \gamma \leq \beta + \delta$.

(5*b) *Further suppose* $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ *and* $\delta \geq 0$. *Then* $\alpha\gamma \leq \beta\delta$.

(6) *Let* $\alpha, \beta \in \mathbb{R}$. *The statements below hold:*

(6*a*) *Suppose* $\alpha\beta > 0$. *Then* ($\alpha > 0$ *and* $\beta > 0$) *or* ($\alpha < 0$ *and* $\beta < 0$).

(6*b*) *Suppose* $\alpha\beta < 0$. *Then* ($\alpha > 0$ *and* $\beta < 0$) *or* ($\alpha < 0$ *and* $\beta > 0$).

(6*) *Let* $\alpha, \beta \in \mathbb{R}$. *The statements below hold:*

(6*a) *Suppose* $\alpha\beta \geq 0$. *Then* ($\alpha \geq 0$ *and* $\beta \geq 0$) *or* ($\alpha \leq 0$ *and* $\beta \leq 0$).

(6*b) *Suppose* $\alpha\beta \leq 0$. *Then* ($\alpha \geq 0$ *and* $\beta \leq 0$) *or* ($\alpha \leq 0$ *and* $\beta \geq 0$).

(7) *Let* $\alpha \in \mathbb{R}$. *Suppose* $\alpha \neq 0$. *Then* $\alpha^2 > 0$.

(7*) *Suppose* $\alpha \in \mathbb{R}$. *Then* $\alpha^2 \geq 0$.

We do not claim that this list is exhaustive in any sense. Nor do we claim that each item in the list is as ‘basic’ as each other.

In fact, some of the above statements are regarded as more ‘basic’ in your *analysis* course. They are the ones listed amongst the Laws of Arithmetic for the reals, and Laws of order for the reals (compatible to the arithmetic operations). The others are deduced from these thirteen statements.

15. **Laws of Arithmetic for the reals.**

(A1) *For any* $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$.

(A2) *For any* $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.

(A3) *There exists some* $z \in \mathbb{R}$, *namely* $z = 0$, *such that for any* $a \in \mathbb{R}$, $a + z = a$ *and* $z + a = a$.

(A4) *For any* $a \in \mathbb{R}$, *there exists some* $b \in \mathbb{R}$, *called an* **additive inverse of** a, *such that* $a + b = 0$ *and* $b + a = 0$.

(A5) *For any* $a, b \in \mathbb{R}$, $a + b = b + a$.

(A6) *For any* $a, b \in \mathbb{R}$, $a \times b \in \mathbb{R}$.

(A7) *For any* $a, b, c \in \mathbb{R}$, $(a \times b) \times c = a \times (b \times c)$.

(A8) *There exists some* $u \in \mathbb{R}$, *namely* $u = 1$, *such that for any* $a \in \mathbb{R}$, $a \times u = a$ *and* $u \times a = a$.

(A9) *For any* $a \in \mathbb{R}\backslash\{0\}$, *there exists some* $b \in \mathbb{R}$, *called a* **multiplicative inverse of** a, *such that* $a \times b = 1$ *and* $b \times a = 1$.

(A10) *For any* $a, b \in \mathbb{R}$, $a \times b = b \times a$.

(A11) *For any* $a, b, c \in \mathbb{R}$, $(a + b) \times c = (a \times c) + (b \times c)$ *and* $a \times (b + c) = (a \times b) + (a \times c)$.

Laws of Order for the reals, compatible with the Laws of Arithmetic.

(O1) *For any* $a, b \in \mathbb{R}$, *if* $a \geq 0$ *and* $b \geq 0$ *then* $a + b \geq 0$ *and* $a \times b \geq 0$.

(O2) *For any* $a \in \mathbb{R}$, $a \geq 0$ *or* $-a \geq 0$.

(O3) *For any* $a \in \mathbb{R}$, *if* $a \geq 0$ *and* $-a \geq 0$ *then* $a = 0$.

16. **Digression on logic: 'Direct proof'.**

Re-examine our work on, say, Statement (C):

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

We have given the passage below as a 'proof' of Statement (C):

Let x, y be positive real numbers. Suppose $x^2 > y^2$.
Then $x^2 - y^2 > 0$.
Note that $x^2 - y^2 = (x - y)(x + y)$. Then $(x - y)(x + y) > 0$.
Therefore $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$.
Since $x > 0$ and $y > 0$, we have $x + y > 0$.
Then $x - y > 0$ and $x + y > 0$.
In particular $x - y > 0$. Therefore $x > y$.

Such a passage is called a 'direct proof' for Statement (C), in the sense that:

- the assumption of Statement (C), namely, '*x, y are positive real numbers and $x^2 > y^2$*', is the starting point of the passage,
- the conclusion of Statement (C), namely, '$x > y$', is the end point of the passage, and
- what is written at each 'intermediate step' will have, as its justification, something already established within the passage, or something known to be 'true in general'.

This is made apparent when we very formally present the passage as the list of statements labelled by roman numerals below:

I. Let x, y be positive real numbers. [Assumption.]
II. Suppose $x^2 > y^2$. [Assumption.]
III. $x^2 - y^2 > 0$. [**II**.]
IV. $x^2 - y^2 = (x - y)(x + y)$. [Properties of the reals.]
V. $(x - y)(x + y) > 0$. [**III**, **IV**.]
VI. ($x - y > 0$ and $x + y > 0$) or ($x - y < 0$ and $x + y < 0$). [**V**, properties of the reals.]
VII. $x + y > 0$ [**I**.]
VIII. $x - y > 0$. [**VI**, **VII**.]
IX. $x > y$. [**VIII**.]

The content of the square bracket at the end of each line indicates the specific immediate reason for the statement in that line. (For example, the immediate reason for 'V' is 'III' and 'IV'.)

The argument for every other statement, apart from Statement (H), in this handout is a 'direct proof'. (Statement (H) is the 'conjunction' of several 'simpler' statements, the argument for each of them is a 'direct proof'.)