MATH1050 Examples: Solving equations and inequalities with algebraic methods.

1. Solve for all real solutions of each of the equations below. 'Check solution' when indeed you have to do so.

(a) $\dfrac{4x-7}{3x+5}=\dfrac{5}{3}$.

(b) $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-8}{x-6}-\dfrac{x-9}{x-7}$.

(c) $\dfrac{x^2-1}{x^2+1}=\dfrac{1}{2}$.

(d) $\dfrac{1}{x^3-x^2-x+1}+\dfrac{1}{x^3-3x^2-x+3}=\dfrac{2}{x^3-x^2-2x}$.

2. Solve for all real solutions of each of the equations below. 'Check solution' when indeed you have to do so.

(a) $\sqrt{2x+9}=x-3$.

(b) $\sqrt{2x-3}=\sqrt{1-2x}$.

(c) $\sqrt{x}-\dfrac{6}{\sqrt{x}}=1$.

(d) $\sqrt{x}-\sqrt{x-2}=1$.

(e) $\sqrt{2x+3}+\sqrt{x+1}=\sqrt{8x+1}$.

(f) $\sqrt{5x+1}+\sqrt{x+1}=\sqrt{10x+6}$.

(g) $\sqrt{x^2+5x+2}=1+\sqrt{x^2+5}$.

(h) $\dfrac{\sqrt{x}+9}{\sqrt{x}-6}=\dfrac{\sqrt{x}-5}{\sqrt{x}-13}$.

(i) $\dfrac{1}{\sqrt{x^2-1}-x}+\dfrac{1}{\sqrt{x^2-1}+x}=-8$.

3. Solve for all real solutions of each of the equations below. 'Check solution' when indeed you have to do so.

(a) $3^{2x+1}-25\cdot 3^x-18=0$.

(b) $5^{x+1}+4\cdot 5^{1-x}=25$.

(c) $2^{(x^2-1)}\cdot 3^{2x-3}=24$.

(d) $\ln(x)+\ln(2x-1)=0$.

(e) $\log_{10}(x^2+1)-\log_{10}(x-2)=1$.

(f) $\log_2(x)-\log_x(8)=2$.

(g) $\log_{10}(x^2+9)-2\log_{10}(x)=1$.

(h) $\log_2(x+1)+\log_2(x+4)=1+2\log_2(3)$.

(i) $\log_3(\log_2(x))+2\log_9(\log_7(8))=2$.

(j) $(\ln(x))^2=\ln(x^2)$.

(k) $2\ln(x^{\ln(x)})+5\ln(x)=3$.

4. Solve for all real solutions of each of the equations below. 'Check solution' when indeed you have to do so.

(a) $|3x-5|=31$.

(b) $3|x-2|=10$.

(c) $|2-1/x|=3$.

(d) $|x^2-5x|=6$.

(e) $|x^2+x-13|=7$.

(f) $|x^2-5x+2|=2$.

(g) $2x=|x-2|$.

(h) $|x-1|=|x^2-4x+3|$.

(i) $|x-3|=|x^2-4x+3|$.

(j) $|x-1|=|x|-1$.

(k) $|x^2-x-8|=|4x-2|$.

(l) $|x^2-4|=x-2$.

(m) $(x-3)^2-|x-3|-12=0$.

(n) $(x-5)^2-2|x-5|-8=0$.

(o) $(x-1)|x|=x|x-1|$.

5. Consider each of the equations below. Determine whether it has any real solution at all. Where it does, determine all its real solutions. Justify your answer.

(a) $x=x$.

(b) $0\cdot x=0$.

(c) $\dfrac{x^2-2x+1}{x-1}=0$.

(d) $\dfrac{x}{x-1}=\dfrac{1}{x-1}$.

(e) $\dfrac{x^2-1}{x-1}=0$.

(f) $\dfrac{x}{x}=1$.

(g) $\dfrac{1}{x-1}=\dfrac{1}{x-1}$.

(h) $\dfrac{1}{x-1}=\dfrac{x+1}{x^2-1}$.

6. Solve for all real solutions of each of the systems of equations below. 'Check solution' when indeed you have to do so.

(a) $\begin{cases} 3x + 2y = 5 \\ x^2 - 4xy + 3 = 0 \end{cases}$

(b) $\begin{cases} 3x^2 - xy - y^2 = 3 \\ x + y = 9 \end{cases}$

(c) $\begin{cases} 2x^2 - y^2 = 2y \\ 6x^2 + xy - y^2 = 8y \end{cases}$

(d) $\begin{cases} x^2 - xy - y^2 = y \\ x^2 - 4y^2 = 0 \end{cases}$

(e) $\begin{cases} 1/x^2 + 1/y^2 = 34 \\ 15xy = 1 \end{cases}$

(f) $\begin{cases} x^2 + y^2 = 5 \\ 1/x^2 + 1/y^2 = 5/4 \end{cases}$

(g) $\begin{cases} x/y + y/x = 17/4 \\ x^2 - 4xy + y^2 = 1 \end{cases}$

(h) $\begin{cases} x - y = 3 \\ \log_{10}(x) + \log_{10}(y) = 1 \end{cases}$

Remark. At some stage of the calculation, brute force is necessary. However, try to observe before starting any calculation how you may simplify a system before resorting to brute force.

7. Solve for all real solutions of each of the inequalities (or systems of inequalities) below.

(a) $x^2 \geq 5x - 6$.

(b) $(x-2)(x+3) < 2(x-2)$.

(c) $(x+8)(2x-3) < (x-5)(x+8)$.

(d) $(x-1)(x-2)(x-3) \geq 27x - 6$.

(e) $(x-1)^2(x-4) \geq 0$.

(f) $(x-1)(x-3)^2 \leq 0$.

(g) $(x+3)x(x-1)(x-2) > 0$.

(h) $(x-1)(x-2)(x-4)(x-8) \leq 0$.

8. Solve for all real solutions of each of the inequalities (or systems of inequalities) below.

(a) $x > -\dfrac{5}{x} + 6$.

(b) $x \leq -\dfrac{6}{x+1} + 4$.

(c) $2x - 1 \leq \dfrac{3}{x-1} - 4$.

(d) $\dfrac{2x}{x+1} \geq 2x - 1$.

(e) $\dfrac{2x-3}{x+1} \leq 1$.

(f) $\dfrac{3x+1}{x+2} \geq 1$.

(g) $\dfrac{1}{x+1} \leq \dfrac{1}{3-x}$.

(h) $\dfrac{1}{x^2-6x+8} \geq 0$.

(i) $\dfrac{3}{x^2-6x+8} \geq 1$.

(j) $\dfrac{x^2-7x+12}{x^2-3x+2} \leq 0$.

(k) $\dfrac{x^2-7x+12}{x^2-3x+2} \leq -1$.

(l) $\dfrac{x^2-1}{x^2-4} \geq 0$.

(m) $\dfrac{x^2-1}{x^2-4} \geq 1$.

9. Solve for all real solutions of each of the inequalities (or systems of inequalities) below:

(a) $|x+3| < 2$.

(b) $|2x-9| \leq 15$.

(c) $|8-3x| \leq 7$.

(d) $|x-2| > 4$.

(e) $|2x+5| \geq 13$.

(f) $|6-x| \geq 6$.

(g) $|x^2+7x-1| < 7$.

(h) $|2x^2-8x-1| \leq 9$.

(i) $|-x^2+2x+3| \geq 5$.

(j) $|x^2-x-3| < 3$.

(k) $\left|\dfrac{3x-1}{4x+1}\right| > 0$.

(l) $\big|\ 2|x| - 9\ \big| \leq 5$.

(m) $x^2 < |x+2|$.

(n) $|3x+1| \geq x^2 + 1$.

(o) $\dfrac{|x-3|}{2x} < 1$.

(p) $\dfrac{|x-9|}{3x+1} > 1$.

(q) $|4x+1| > |x-3|$.

(r) $(x+2)|x-2| < -5$.

(s) $x^2 - |x| - x < 0$.

10. Solve for all real solutions of the inequalities below:

    (a) $\sqrt{4x+1} < x+1$.

    (b) $\sqrt{6x+3} > 3x+1$.

    Remark. The absolute value is implicitly involved in these inequalities: what do you obtain when you square both sides of each inequalities?

11. Let c be a real number. Consider the equation

$$cx = c + 1 \quad \text{——} \quad (\star_c)$$

    with unknown x.

    (a) Suppose $c \neq 0$. Write down all real solutions of $(\star_c)$.

    (b) Suppose $c = 0$. Does $(\star_c)$ have any real solution? Justify your answer.

12. Let c be a real number. Consider the equation

$$cx = c(c+1) \quad \text{——} \quad (\star_c)$$

    with unknown x.

    (a) Suppose $c \neq 0$. Write down all real solutions of $(\star_c)$.

    (b) Suppose $c = 0$. Does $(\star_c)$ have any real solution? Justify your answer.

13. Let a, b be real numbers. Consider the equation

$$(a^2 - 4a + 3)x = b - 2 \quad \text{——} \quad (\star_{a,b})$$

    with unknown x.

    (a) Suppose $a^2 - 4a + 3 \neq 0$. Write down all real solutions of $(\star_{a,b})$.

    (b) Suppose $a^2 - 4a + 3 = 0$.

        i. Suppose $(\star_{a,b})$ has a real solution. What are the respective values of a, b?

        ii. Determine all the solutions of $(\star_{a,b})$ where it has any solution at all.

14. Let c be a real number. Consider the equation

$$\ln(x + c) = \ln(c) + \ln(x) \quad \text{——} \quad (\star_c)$$

    with unknown x.

    (a) Suppose $(\star_c)$ has a real solution. Find all real solutions of $(\star_c)$.

    (b) For which values of c does $(\star_c)$ have any real solution? Justify your answer.

15. Let c be a real number. Consider the system of equations

$$(\star_c) \begin{cases} x + 2y = 3 \\ 2x + 3y = 4 \\ 3x + cy = 5 \end{cases}$$

    with unknowns x, y.

    (a) Suppose $(\star_c)$ has a real solution. Find all possible value(s) of c.

    (b) For each such values of c, solve $(\star_c)$.