MATH1050 Examples: Arithmetic progressions and geometric progressions.

1. Let a, b be non-zero numbers. Suppose $a, b, 1$ are in geometric progression and $a, b+1, 1$ are in arithmetic progression. Prove that $a, b+1, 2$ are in geometric progression.

2. Let a, b, c be real numbers. Suppose $c-a, c, c+b$ are non-zero, and are in geometric progression.
   (a) Prove that $(Pa+Qb)c = ab$. Here P, Q are integers whose values you have to find explicitly.
   (b) Suppose $a = b$. Find the value of a, b.
   (c) Suppose $a \neq b$. Express c in terms of a, b.

3. Let a, b be non-zero real numbers. Suppose $a, \frac{1}{4}, b$ be in geometric progression, and $\frac{1}{a}, 5, \frac{1}{b}$ be in arithmetic progression. Find all possible values of a, b.

4. Let $a_0, a_1, a_2, a_3, \cdots$ be a geometric progression of real numbers. Suppose $a_2 = -24$ and $a_5 = 81$ respectively.
   (a) Find the value of a_0.
   (b) Prove that $a_n = \frac{(-1)^{n-P} B^{Qn-R}}{C^{n-S}}$ for each n. Here B, C, P, Q, R, S are some appropriate positive integers, independent of n, whose values you have to determine explicitly.

5. Let a be a non-zero real number. Suppose $-a, 2a, 3a^2, b$ are in geometric progression.
   (a) Find the value of a.
   (b) Find the value of b.

6. Let a, b, c, d be positive real numbers. Suppose a, b, c, d are in geometric progression.

   Prove that $\sqrt{\frac{a^5+b^2c^2+a^3c^2}{b^4c+d^4+b^2cd^2}} = M\left(\frac{a}{b}\right)^N$. Here M, N are integers whose respective values you have to determined explicitly.

7. Let $a_1, a_2, a_3, \cdots$ be in arithmetic progression. Suppose $a_1+a_2+a_3 = 3$ and $a_1a_2a_3 = -24$. Find the value of a_n for each positive integer n.

8. Suppose the numbers a, b, c form an arithmetic progression.
   (a) Suppose a, b, c also form a geometric progression. Prove that $a = b = c$.
   (b) Now instead suppose a, c, b form a geometric progression. Further suppose that $a+b+c = 6$. Find the values of a, b, c respectively.

9. Let a, b, c be non-zero numbers.

   Suppose $a+b, b+c, c+a$ are all non-zero. Suppose $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in arithmetic progression.

   Prove that $\frac{bc}{b+c}, \frac{ca}{c+a}, \frac{ab}{a+b}$ are in arithmetic progression.

10. Let $s, t, u, v \in \mathbb{C}\backslash\{0\}$.

    Suppose $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}, \frac{1}{v}$ form an arithmetic progression.
    (a) Prove that $t = \frac{2su}{s+u}$.
    (b) Prove that $\frac{3s-u}{s+u}$.

11. Let $u, v, w \in \mathbb{C}\backslash\{0\}$.

    Suppose they are pairwise distinct. Suppose $\frac{1}{u}, \frac{1}{v}, \frac{1}{w}$ form an arithmetic progression. Also suppose u, w, v form a geometric progression.

(a) Prove that $w = -2u$.

(b) Hence, or otherwise, prove that v, u, w form an arithmetic progression.

12. Let $a, b, c, d \in \mathbb{C}\backslash\{0\}$.

Suppose $\frac{a}{b} = \frac{c}{d}$. Further suppose that a, b, c form an arithmetic progression.

Prove that $\frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ form an arithmetic progression.

13. (a) Let $k \in \mathbb{N}\backslash\{0, 1\}$.
Suppose $b_0, b_1, b_2, \cdots, b_k$ form arithmetic progression. Further suppose $b_0 + b_1 + b_2 + \cdots + b_k = 0$.
Prove that $b_j + b_{k-j} = 0$ for each $j \in [\![0, k]\!]$.

(b) Let $\{c_p\}_{p=0}^{\infty}$ be an arithmetic progression.
Let $m, n \in \mathbb{N}$. Suppose $m < n$. Suppose $c_0 + c_1 + c_2 + \cdots + c_m = c_0 + c_1 + c_2 + \cdots + c_n$.
Prove that $c_0 + c_1 + c_2 + \cdots + c_{m+n+1} = 0$.

14. Let $c_0, c_1, c_2, \cdots, c_n \in \mathbb{C}\backslash\{0\}$. Suppose $c_0, c_1, c_2, \cdots, c_n$ are in geometric progression.

Define $S = c_0 + c_1 + c_2 + \cdots + c_n$, $P = c_0 \cdot c_1 \cdot c_2 \cdot ... \cdot c_n$, and $R = \frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}$.

Prove that $\left(\frac{S}{R}\right)^{n+1} = P^2$.

15. Let c be a positive real number. Let α, β, γ be real numbers, with $\alpha < \beta < \gamma$.

Suppose α, β, γ are in arithmetic progression. Further suppose $\sin(c\alpha) \neq 0$, and $\sin(c\alpha), \sin(c\beta), \sin(c\gamma)$ are in geometric progression.

Prove that the smallest possible value of $\beta - \alpha$ is $\frac{\pi}{c}$.

16. Let $a_1, a_2, a_3, \cdots$ be in arithmetic progression, with common difference d. Suppose $a_1 \neq 0$ and $d \neq 0$. For each positive integer n, denote by s_n the sum of $a_1, a_2, \cdots, a_n$.

(a) Prove that $\frac{s_{2n} - s_n}{s_n} = \frac{Aa_1 + (Bn - 1)d}{Aa_1 + (n - 1)d}$ for each positive integer n. Here A, B are positive integers whose values are independent of n, which you have to determine explicitly.

(b) Suppose $\frac{s_4 - s_2}{s_2} = \frac{s_6 - s_3}{s_3}$.

i. Express d in terms of a_1.

ii. Prove that $\frac{s_{2n} - s_n}{s_n}$ is constant for all values of n. What is the value of this constant?

17. (a) Let $m \in \mathbb{N}$. Let s be any number which is not equal to 1. Verify that

$$1 + 2s + 3s^2 + \cdots + ms^{m-1} + (m + 1)s^m = \frac{A - (m + 2)s^{m+B} + (m + 1)s^{m+C}}{(1 - s)^D},$$

where A, B, C, D are integers independent of the value of s. You have to determine the respective values of A, B, C, D explicitly.

(b) Let $n \in \mathbb{N}$. Let a, b be any real numbers. Prove that

$$(a - b)^E[a^m + 2a^{m-1}b + 3a^{m-2}b^2 + \cdots + mab^{m-1} + (m + 1)b^m] = a^{m+F} - (m + G)ab^{m+H} + (m + J)b^{m+K}.$$

Here E, F, G, H, J, K are integers independent of the value of m. You have to determine the respective values of E, F, G, H, J, K explicitly.