MATH1050 Examples: Quadratic polynomials and quadratic functions.

1. Let c be a real number, and $f(x)$ be the polynomial $(3x+2)(2x-3)-(3c+2)(2c-3)$.

   Find the roots of $f(x)$.

2. Let c be a real number.

   Let $f(x)$ be the polynomial $2x^2+(c+2)x-2(c+3)$, and α, β be the roots of $f(x)$. Suppose $\alpha+\beta=\alpha\beta$.

   Find the value of c.

3. Let $f(x)$ be the polynomial x^2+5x-4, and α, β be the roots of $f(x)$.

   Without determining the values of α, β explicitly, find the values of the expressions below:

   (a) $\alpha^2+\beta^2$
   (b) $\alpha^4+\beta^4$
   (c) $\alpha^3+\beta^3$
   (d) $\beta^{-1}(2-\alpha)+\alpha^{-1}(2-\beta)$
   (e) $\sqrt{(\alpha-\beta)^2}$
   (f) $\sqrt{(\alpha^2-\beta^2)^2}$
   (g) $2^\alpha \cdot 2^\beta$
   (h) $\log_4(\alpha^2)+\log_4(\beta^2)$

4. Let a, b, c be numbers, with $a \neq 0$, and $f(x)$ be the quadratic polynomial given by $f(x)=ax^2+bx+c$.

   Suppose that α, β are the roots of $f(x)$.

   (a) i. Find the quadratic polynomial with leading coefficient a whose roots are $3\alpha-1$ and $3\beta-1$.
   ii. Find the quadratic polynomial with leading coefficient a^2 whose roots are α^2 and β^2.
   iii. Find the quadratic polynomial with leading coefficient a^2 whose roots are $\alpha+3\beta$ and $\beta+3\alpha$.

   (b) Now further suppose $c \neq 0$.
   i. Find the quadratic polynomial with leading coefficient ac whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
   ii. Find the quadratic polynomial with leading coefficient ac whose roots are $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.
   iii. Find the quadratic polynomial with leading coefficient a^2c^2 whose roots are $\alpha^2+\frac{1}{\beta^2}$ and $\beta^2+\frac{1}{\alpha^2}$.

5. Let a, b be real numbers.

   Let $f(x), g(x)$ be the quadratic polynomials given by $f(x)=-x^2+ax+1$, $g(x)=x^2+bx+a$ respectively.

   Suppose α, β are the roots of $f(x)$ and $2\alpha, 2\beta$ are the roots of $g(x)$.

   Determine the values of a, b respectively.

6. Let a, b, c, p, q, r be numbers, with $a \neq 0$ and $p \neq 0$.

   Let $f(x), g(x)$ be the quadratic polynomials given by $f(x)=ax^2+bx+c$, $g(x)=px^2+qx+r$ respectively.

   Suppose α, β are the roots of $f(x)$, and $\alpha \neq 0$, $\beta \neq 0$. Further suppose $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $g(x)$.

   Prove that $ap=cr$ and $aq=br$.

7. Let p, q, r be numbers. Suppose $p+q-2r \neq 0$.

   Let $f(x)$ be the quadratic polynomial given by $f(x)=(p+q-2r)x^2+(q+r-2p)x+(r+p-2q)$.

   (a) Verify that 1 is a root of $f(x)$.
   (b) Prove that $f(x)$ has a repeated root iff $q=r$.

8. Let k be a number, and $f(x)$ be the quadratic polynomial given by $f(x)=x^2-2x+k$.

   Suppose α, β are the roots of $f(x)$.

   (a) Find the quadratic polynomial $g(x)$ with leading coefficient 1 whose roots are α^3, β^3.
   (b) Also prove that the discriminant Δ_g of $g(x)$ is given by $\Delta_g=A(B-k)(C-k)^2$. Here A, B, C are integers, whose values you have to determine explicitly.
   (c) Suppose k is a real number. Further suppose α, β are not real numbers, but α^3, β^3 are real numbers.
   Find all possible values of k. Justify your answer.

9. Let a, b be real numbers. Suppose $a > b$ and $a + b \neq 0$.

   Let $f(x)$ be the quadratic polynomial given by $f(x) = (a - b)x^2 - 2(a^2 + b^2)x + (a^3 - b^3)$.

   (a) Prove that the roots of $f(x)$ are real and distinct iff $ab > 0$.

   (b) Suppose α, β are distinct real roots of $f(x)$, and $\alpha > \beta$.

   Prove that $\alpha - \beta = \dfrac{2(a+b)\sqrt{ab}}{a-b}$.

10. Let a, b be real numbers. Suppose $a > b > 0$.

    Let $f(x)$ be the quadratic polynomial given by $f(x) = 2x^2 - (3a + b)x + ab$.

    Prove that $f(x)$ has two distinct real roots, one of them greater than b and the other less than b.

11. Let p be a real number. Let $f(x)$ be the quadratic polynomial given by $f(x) = x^2 + (p + 1)x + (p - 1)$.

    Suppose α, β are the roots of $f(x)$.

    (a) Prove that α, β are real and distinct.

    (b) Express $(\alpha - 2)(\beta - 2)$ in terms of p.

    (c) Suppose $\beta < 2 < \alpha$.

    i. Prove that $p < -\dfrac{5}{3}$.

    ii. Further suppose $(\alpha - \beta)^2 < 20$. Prove that $-3 < p < -\dfrac{5}{3}$.

12. Let c be a real number. Let $f(x)$ be the polynomial given by $f(x) = (c - 4)x^2 + (2c - 1)x + (4c - 1)$.

    Suppose α, β are the roots of $f(x)$, and $\alpha < 0 < \beta$.

    (a) By considering the product $\alpha\beta$, or otherwise, prove that $\dfrac{1}{4} < c < 4$.

    (b) Further suppose $\alpha + \beta < 0$. Prove that $\dfrac{1}{4} < c < \dfrac{1}{2}$.

13. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 - 2x + 3}{x^2 + 2x + 3}$ for any $x \in \mathbb{R}$.

    (a) Let $\alpha \in \mathbb{R}$. Prove that $2 - \sqrt{3} \leq f(\alpha) \leq 2 + \sqrt{3}$.

    Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $f(\alpha) = \dfrac{\alpha^2 - 2\alpha + 3}{\alpha^2 + 2\alpha + 3}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

    (b) Prove that f attains absolute minimum value $2 - \sqrt{3}$ and attains absolute maximum value $2 + \sqrt{3}$.

14. Let $f : \mathbb{R}\backslash\{-1\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 + x + 1}{x + 1}$ for any $x \in \mathbb{R}\backslash\{-1\}$.

    (a) Let $\alpha \in \mathbb{R}\backslash\{-1\}$. Prove that $f(\alpha) \leq -3$ or $f(\alpha) \geq 1$.

    Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $\beta = \dfrac{\alpha^2 + \alpha + 1}{\alpha + 1}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

    (b) Does f attain the values -3, 1? Justify your answer.

15. Let $f : \mathbb{R}\backslash\{2, 4\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{(x - 1)(x - 5)}{(x - 2)(x - 4)}$ for any $x \in \mathbb{R}\backslash\{2, 4\}$.

    (a) Let $\alpha \in \mathbb{R}\backslash\{2, 4\}$. Prove that $f(\alpha) \leq 1$ or $f(\alpha) \geq 4$.

    Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $\beta = \dfrac{(\alpha - 1)(\alpha - 5)}{(\alpha - 2)(\alpha - 4)}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

    (b) Does f attain the value 1? Justify your answer.

    (c) Does f attain the value 4? Justify your answer.