MATH1050 Arithmetic progression and geometric progression

1. **Definition. (Arithmetic progression.)**

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be an **arithmetic progression** *if the statement (AP) holds:*

   (AP) *There exists some $d \in \mathbb{C}$ such that for any $n \in \mathbb{N}$, $a_{n+1} - a_n = d$.*

   The number d is called the **common difference** *of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$.*

   Remark. The use of the article '*the*' should be justified with a 'uniqueness' result, which is Lemma (1).

   Remark on 'terminating arithmetic progressions'. Let $N \in \mathbb{N}\backslash\{0,1\}$. Suppose $c_0, c_1, \cdots, c_N$ are $N+1$ complex numbers. We abuse notation in saying that $c_0, c_1, \cdots, c_N$ form an arithmetic progression with common difference d exactly when there exists some arithmetic progression $\{a_n\}_{n=0}^{\infty}$ with common difference d such that $a_k = c_k$ for any integer k amongst $0, 1, 2, \cdots, N$. (In plain language, $c_0, c_1, \cdots, c_N$ are identified as the 0-th term, 1-st term, ... , N-th term of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$.)

2. **Lemma (1).**

   Let $\{a_n\}_{n=0}^{\infty}$ be an arithmetic progression, and d, d' be complex numbers. Suppose d, d' are common differences of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$. Then $d = d'$.

   Remark. This is how we formulate the statement '*each arithmetic progression has at most one common difference*'.

   Proof of Lemma (1).

   Let $\{a_n\}_{n=0}^{\infty}$ be an arithmetic progression, and d, d' be complex numbers. Suppose d, d' are common differences of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$.

   By definition, we have $a_1 - a_0 = d$.

   Also, by definition, we have $a_1 - a_0 = d'$.

   Then $d = d'$.

3. **Lemma (2).**

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression, with common difference d.

   Then $a_{m+k} = a_m + kd$ for any $m, k \in \mathbb{N}$. (In particular $a_n = a_0 + nd$ for any $n \in \mathbb{N}$.)

   Proof of Lemma (2). Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression, with common difference d.

   Pick any $m, k \in \mathbb{N}$. By definition, these k equalities hold:

$$\left\{\begin{array}{lcrcl} a_{m+k} & - & a_{m+k-1} & = & d \\ a_{m+k-1} & - & a_{m+k-2} & = & d \\ & & & \vdots & \\ a_{m+1} & - & a_m & = & d \end{array}\right.$$

   Then $a_{m+k} - a_m = \displaystyle\sum_{j=1}^{k}(a_{m+j} - a_{m+j-1}) = \sum_{j=1}^{k} d = kd$.

   Therefore $a_{m+k} = a_m + kd$.

4. **Lemma (3). ('Converse' of Lemma (2).)**

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose $a_{m+k} = a_m + kd$ for any $m, k \in \mathbb{N}$.

   Then $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression, with common difference d.

   Proof of Lemma (3).

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose $a_{m+k} = a_m + kd$ for any $m, k \in \mathbb{N}$.

   Then, in particular, $a_k = a_0 + kd$ for any $k \in \mathbb{N}$.

   Pick any $n \in \mathbb{N}$. We have $a_{n+1} - a_n = [a_0 + (n+1)d] - (a_0 + nd) = d$.

It follows that $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression, with common difference d.

Reminder on terminology in logic. Compare the respective statements of Lemma (2) and Lemma (3). The latter is called the converse of the former because the respective positions of

- '$\{a_n\}_{n=0}^{\infty}$ *is an arithmetic progression, with common difference* d',
- '$a_{m+k} = a_m + kd$ *for any* $m, k \in \mathbb{N}$'

have been interchanged.

5. **Lemma (4).**

   Let $\{a_n\}_{n=0}^{\infty}$ *be an infinite sequence in* $\mathbb{C}$. *Suppose* $\{a_n\}_{n=0}^{\infty}$ *is an arithmetic progression.*

   Then for any $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

   Proof of Lemma (4). Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression.
   Pick any $k \in \mathbb{N}$. By definition, $a_{k+1} - a_k = a_{k+2} - a_{k+1}$.

   Then $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

6. **Lemma (5). ('Converse' of Lemma (4))**

   Let $\{a_n\}_{n=0}^{\infty}$ *be an infinite sequence in* $\mathbb{C}$. *Suppose for any* $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

   Then $\{a_n\}_{n=0}^{\infty}$ *is an arithmetic progression.*

   Proof of Lemma (5). Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Suppose for any $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

   Define $d = a_1 - a_0$.
   Pick any $m \in \mathbb{N}$. (If $m = 1$ then by definition, $a_{m+1} - a_m = d$.) Suppose $m \geq 1$. Then by assumption, these equalities hold:

$$\left\{\begin{array}{ccccccc} a_{m+1} & - & a_m & = & a_m & - & a_{m-1} \\ a_m & - & a_{m-1} & = & a_{m-1} & - & a_{m-2} \\ & & & \vdots & & & \\ a_3 & - & a_2 & = & a_2 & - & a_1 \\ a_2 & - & a_1 & = & a_1 & - & a_0 \end{array}\right.$$

   Therefore $a_{m+1} - a_m = a_1 - a_0 = d$.

   Hence by definition, $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression.

7. We can summarize what we have learnt in Lemma (2), Lemma (3), Lemma (4), Lemma (5) into the result below.

   Theorem (6). (Equivalent formulations of the definition of arithmetic progression.)

   Let $\{a_n\}_{n=0}^{\infty}$ *be an infinite sequence in* $\mathbb{C}$. *The statements below are logically equivalent:*

   (a) $\{a_n\}_{n=0}^{\infty}$ *is an arithmetic progression, with common difference* d.
   (b) *For any* $m, k \in \mathbb{N}$, $a_{m+k} = a_m + kd$.
   (c) *For any* $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

8. **Lemma (7).**

   Suppose $n \in \mathbb{N}$. *Then the equality* $0 + 1 + 2 + 3 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$ *holds.*

   Proof of Lemma (7). Suppose $n \in \mathbb{N}$. Write $S_n = 0 + 1 + 2 + 3 + \cdots + (n-1) + n$.

   By definition, $S_n = n + (n-1) + (n-2) + (n-3) + \cdots + 1 + 0$ also.

   Then

$$\begin{aligned} 2S_n &= S_n + S_n = [0 + 1 + 2 + 3 + \cdots + (n-1) + n] + [n + (n-1) + (n-2) + (n-3) + \cdots + 1 + 0] \\ &= (0 + n) + [1 + (n-1)] + [2 + (n-2)] + \cdots + [(n-1) + 1] + (n + 0) \\ &= \underbrace{n + n + n + \cdots + n + n}_{n+1 \text{ times}} = (n+1)n. \end{aligned}$$

Therefore $S_n = \dfrac{n(n+1)}{2}$.

9. **Theorem (8). (Formula for the sum of finitely many consecutive terms in an arithmetic progression.)**

   Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression with common difference d.

   Then, for any $m, k \in \mathbb{N}$, $a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+k} = (k+1)a_m + \dfrac{k(k+1)}{2}d$.

   Proof of Theorem (8). Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression with common difference d.

   Pick any $m, k \in \mathbb{N}$. By Theorem (6), we have $a_{m+j} = a_m + jd$ for each $j = 0, 1, 2, \cdots, k$.

   Then

$$\begin{aligned} a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+k} &= a_m + (a_m + d) + (a_m + 2d) + \cdots + (a_m + kd) \\ &= (k+1)a_m + (0 + 1 + 2 + \cdots + k)d \\ &= (k+1)a_m + \frac{k(k+1)}{2}d. \end{aligned}$$

10. **Definition. (Geometric progression.)**

    Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}\backslash\{0\}$. The infinite sequence $\{b_n\}_{n=0}^{\infty}$ is said to be a **geometric progression** *if the statement (GP) holds:*

    (GP) *There exists some $r \in \mathbb{C}\backslash\{0\}$ such that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$.*

    The number r is called the **common ratio** *of this geometric progression.*

    Remark. The use of the article '*the*' should be justified with a 'uniqueness' result, which is Lemma (9).

    Remark on 'terminating geometric progression'. Let $N \in \mathbb{N}\backslash\{0, 1\}$. Suppose $c_0, c_1, \cdots, c_N$ are $N+1$ non-zero complex numbers. We abuse notation in saying that $c_0, c_1, \cdots, c_N$ form a geometric progression with common ratio r exactly when there exists some geometric progression $\{b_n\}_{n=0}^{\infty}$ with common ratio r such that $b_k = c_k$ for any integer k amongst $0, 1, 2, \cdots, N$. (In plain language, $c_0, c_1, \cdots, c_N$ are identified as the 0-th term, 1-st term, ... , N-th term of the geometric progression $\{b_n\}_{n=0}^{\infty}$.)

11. **Lemma (9).**

    Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression, and r, r' be complex numbers. Suppose r, r' are common ratios of the geometric progression $\{b_n\}_{n=0}^{\infty}$. Then $r = r'$.

    Proof of Lemma (9). Exercise. (Imitate the argument for Lemma (1).)

12. Imitating the reasoning in Lemma (2), Lemma (3), Lemma (4), Lemma (5), we obtain the result below.

    Theorem (10). (Equivalent formulations of the definition of geometric progression.)

    Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}\backslash\{0\}$. The statements below are logically equivalent:

    (a) *$\{b_n\}_{n=0}^{\infty}$ is a geometric progression with common ratio r.*

    (b) *For any $m, k \in \mathbb{N}$, $b_{m+k} = b_0 r^k$.*

    (c) *For any $k \in \mathbb{N}$, $b_{k+1}^2 = b_k b_{k+2}$.*

13. **Lemma (11).**

    Suppose $n \in \mathbb{N}$ and $r \in \mathbb{C}$. Then the statements below hold:

    (a) $1 - r^{n+1} = (1-r)(1 + r + r^2 + \cdots + r^n)$.

    (b) *Further suppose $r \neq 1$. Then $\dfrac{1 - r^{n+1}}{1-r} = 1 + r + r^2 + \cdots + r^n$.*

    Proof of Lemma (11). Suppose $n \in \mathbb{N}$ and $r \in \mathbb{C}$. Write $T_{n,r} = 1 + r + r^2 + \cdots + r^{n-1} + r^n$.

    (a) We have $rT_{n,r} = r + r^2 + \cdots + r^{n-1} + r^n + r^{n+1}$.

    Then $(1-r)T_{n,r} = T_{n,r} - rT_{n,r} = (1 + r + r^2 + \cdots + r^{n-1} + r^n) - (r + r^2 + \cdots + r^{n-1} + r^n + r^{n+1}) = 1 - r^{n+1}$.

(b) Further suppose $r \neq 1$. Then $1 - r \neq 0$. Therefore $\dfrac{1 - r^{n+1}}{1 - r} = \dfrac{(1 - r)T_{n,r}}{1 - r} = T_{n,r} = 1 + r + r^2 + \cdots + r^n$.

14. **Theorem (12).**

Suppose $n \in \mathbb{N}$, and $s, t \in \mathbb{C}$.

Then the equality $s^{n+1} - t^{n+1} = (s - t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n)$ holds.

Proof of Theorem (12). Exercise.

Remark. When we want to 'factorizing' the expression $s^n - t^n$ or $s^n + t^n$ with the help of integers only, we have these equalities below for 'small values' of n:

$$
\begin{array}{ll}
s^2 - t^2 = (s - t)(s + t), & \\
s^3 - t^3 = (s - t)(s^2 + st + t^2), & s^3 + t^3 = (s + t)(s^2 - st + t^2), \\
s^4 - t^4 = (s - t)(s + t)(s^2 + t^2), & \\
s^5 - t^5 = (s - t)(s^4 + s^3t + s^2t^2 + st^3 + t^4), & s^5 + t^5 = (s + t)(s^4 - s^3t + s^2t^2 - st^3 + t^4), \\
s^6 - t^6 = (s - t)(s + t)(s^2 + st + t^2)(s^2 - st + t^2), & s^6 + t^6 = (s^2 + t^2)(s^4 - s^2t^2 + t^2),
\end{array}
$$

Of course, when we resort to roots of unity, we may 'completely factorize' $s^n - t^n$ into a product of 'linear expressions' with the help of complex numbers. Or we may 'factorize' $s^n - t^n$ into a product of 'linear expressions' and 'quadratic expressions' with the help of real numbers only. (We need De Moivre's Theorem and results about roots of unity here.)

15. **Theorem (13). (Formula for the sum of finitely many consecutive terms in a geometric progression.)**

Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression with common ratio r. Then for each $m, n \in \mathbb{N}$,

$$
b_m + b_{m+1} + b_{m+2} + \cdots + b_{m+n} = \begin{cases} (n+1)b_m & \text{if } \; r = 1 \\ \dfrac{b_m(r^{n+1} - 1)}{r - 1} & \text{if } \; r \neq 1 \end{cases}
$$

Proof of Theorem (13). Exercise. (Apply Lemma (10).)

16. **Theorem (14). (Limiting value for a sum of geometric progression.)**

Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression of non-zero real numbers, with common ratio r. Suppose $|r| < 1$.

Then $\lim\limits_{n \to \infty} (b_0 + b_1 + b_2 + \cdots + b_n) = \dfrac{b_0}{1 - r}$.

Remark on the proof of Theorem (14). Apply Theorem (13), and standard techniques in *calculus* such as the Sandwich Rule.

Further remark. In school mathematics, you were concerned with the situation in which r was real (as stated here). But this result can be extended to the situation in which r is a general complex number with modulus less than 1.