MATH1050 Binomial coefficients and binomial expansions

1. **Definition. (Factorials.)**

   For each $n \in \mathbb{N}\backslash\{0\}$, we define the number $n!$ by $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.

   We define $0!$ to be the number 1.

   We refer to the number $m!$ as **factorial** *m.*

2. **Definition. (Binomial coefficients.)**

   Let $n \in \mathbb{N}$. For each integer k, define the number $\begin{pmatrix} n \\ k \end{pmatrix}$ by

   $$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{cases} \dfrac{n!}{k! \cdot (n-k)!} & \text{if } k \text{ is an integer between } 0 \text{ and } n, \\ 0 & \text{if } k \text{ is an integer greater than } n \text{ or less than } 0. \end{cases}$$

   The number $\begin{pmatrix} n \\ k \end{pmatrix}$ is called the **binomial coefficient of n over k**.

   Remark. It turns out that $\begin{pmatrix} n \\ k \end{pmatrix}$ is an integer, although it is not apparent at first sight.

3. **Theorem (1).**

   (a) *Suppose $n \in \mathbb{N}$, and k is an integer between 0 and n. Then $\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n \\ n-k \end{pmatrix}$.*

   (b) *Suppose $n \in \mathbb{N}$, and k is an integer amongst $0, 1, 2, \cdots, n-1$. Then $\begin{pmatrix} n \\ k \end{pmatrix} + \begin{pmatrix} n \\ k+1 \end{pmatrix} = \begin{pmatrix} n+1 \\ k+1 \end{pmatrix}$.*

   Proof. Exercise on the definition for binomial coefficients.

4. **Binomial Theorem for numbers.**

   Let a, b be any numbers.

   For any $n \in \mathbb{N}$, $(a+b)^n = a^n + \begin{pmatrix} n \\ 1 \end{pmatrix} a^{n-1}b + \begin{pmatrix} n \\ 2 \end{pmatrix} a^{n-2}b^2 + \cdots + \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k}b^k + \cdots + \begin{pmatrix} n \\ n-1 \end{pmatrix} ab^{n-1} + b^n$.

   Proof. Exercise in mathematical induction. (Theorem (1) will be needed at some point of the argument.)

5. **'Polynomial version' of Binomial Theorem.**

   Let $n \in \mathbb{N}$.

   $(1+x)^n = 1 + \begin{pmatrix} n \\ 1 \end{pmatrix} x + \begin{pmatrix} n \\ 2 \end{pmatrix} x^2 + \cdots + \begin{pmatrix} n \\ k \end{pmatrix} x^k + \cdots + \begin{pmatrix} n \\ n-1 \end{pmatrix} x^{n-1} + x^n$ as polynomials.

   Proof. Same as that for Binomial Theorem for numbers.

6. **Appendix. Generalized Binomial Theorem.**

   Let α be a real number which is not amongst the natural numbers.

   For any $k \in \mathbb{N}$, define $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.

   For any real number $t \in (-1, 1)$, the infinite sequence $\left\{ \sum_{k=0}^{n} \begin{pmatrix} \alpha \\ k \end{pmatrix} t^k \right\}_{n=0}^{\infty}$ converges to the limit $(1+t)^\alpha$.

   Proof. Postponed to your *mathematical analysis* course.

   Remark. We may re-formulate this result in terms of 'power functions':

Let $u_\alpha : (-1, +\infty) \longrightarrow \mathbb{R}$ *be the real-valued function of one real variable defined by* $u_\alpha(x) = (1+x)^\alpha$ *for any* $x \in (-1, +\infty)$.

The equality $u_\alpha(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} \alpha \\ k \end{pmatrix} x^k$ *holds for any* $x \in (-1, 1)$.

Expressing the result using the dot-dot-dots instead of the summation symbol, we obtain:

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots+\frac{\alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)}{n!}x^n+\cdots \quad \text{for any } x \in (-1,1).$$

So formally we have 'generalized' the Binomial Theorem to the situation where the 'index' α in $(1+x)^\alpha$ is no longer assumed to be a natural number. But there is a price to pay: the 'right-hand-side' of the equality is no longer a finite sum but the limit of a certain infinite sequence, and the equality is valid, for some 'restricted values of x' only, rather than 'arbitrary values of x'.

Two interesting particular cases of this equality are:

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 \pm \cdots$ for any $x \in (-1, 1)$. (This is as expected.)
- $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \cdots$ for any $x \in (-1, 1)$.

Such equalities provide useful tools of approximations for, say, physicists. More intriguing is the fact that the study on the Binomial Theorem and its generalizations (like these equalities) helped lead Newton, Leibniz and other seventeenth and eighteenth century mathematicians to great discoveries on what we now refer to as the calculus.