1. **Convention on the summation symbol.**

   Suppose m, n are integers.

   (a) *Suppose $m \leq n$. Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n$ are numbers.*

   We agree to write $a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$ as $\sum_{k=m}^{n} a_k$.

   (b) *Suppose $m > n$. Then we agree to read $\sum_{k=m}^{n} b_k$ as 0 no matter what the b_k's are.*

   Remark. The symbol k in the expression $\sum_{k=m}^{n} a_k$ is called the dummy index in this expression.

2. **Illustrations.**

   (a) $1 + 3 + 5 + 7 + \cdots + 99 + 101 = \sum_{k=0}^{50} (2k+1)$.

   (b) $\underbrace{1 - 3 + 5 - 7 + 9 - 11 + \cdots - 99 + 101}_{\text{alternating } +,\ -} = \sum_{k=0}^{50} (-1)^k (2k+1)$.

   (c) $1^4 + 3^4 + 5^4 + 7^4 + 9^4 + \cdots + 99^4 + 101^4 = \sum_{k=0}^{50} (2k+1)^4$.

   (d) $\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 3} + \frac{1}{9 \cdot 4} + \frac{1}{16 \cdot 5} + \frac{1}{25 \cdot 6} + \cdot + \frac{1}{361 \cdot 20} + \frac{1}{400 \cdot 21} = \sum_{k=1}^{20} \frac{1}{k^2(k+1)}$.

   (e) $1 + 2^2 + 4^3 + 8^4 + \cdots + 512^8 + 1024^9 = \sum_{k=0}^{10} 2^{k(k-1)}$.

3. **Theorem (1). (Basic properties of the summation symbol.)**

   Let m, n, p, q be integers. Suppose $m \leq n \leq p$.

   Let $a_m, a_{m+1}, \cdots, a_n, a_{n+1}, \cdots, a_p, b_m, b_{m+1}, \cdots, b_n, b_{n+1}, \cdots, b_p, c$. The statements below hold:

   (a) $\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_j$.

   (b) $\sum_{k=m}^{m} a_k = a_m$.

   (c) $\sum_{k=m}^{n} a_k = \sum_{j=m-q}^{n-q} a_{j+q}$.

   (d) $\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_{m+n-j}$.

   (e) $\sum_{k=m}^{n} c = (n-m+1)c$.

   (f) $\sum_{k=m}^{n} ca_k = c \sum_{k=m}^{n} a_k$.

   (g) $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$.

   (h) $\sum_{k=m}^{p} a_k = \sum_{k=m}^{n} a_k + \sum_{k=n+1}^{p} a_k$.

   Remark. Do not be scared by the symbols. Many of these formulae are just 'short-hand' for what you are very familiar. For instance:

   - '$\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_{m+n-j}$' is '$a_m + a_{m+1} + \cdots + a_{n-1} + a_n = a_n + a_{n-1} + \cdots + a_{m+1} + a_m$' in disguise.
   - '$\sum_{k=m}^{n} ca_k = c \sum_{k=m}^{n} a_k$' is '$ca_m + ca_{m+1} + \cdots ca_n = c(a_m + a_{m+1} + \cdots + a_n)$' in disguise.

   Proof. Apply mathematical induction. (Hence they are postponed at this moment but will be left to you).

4. **Some basic results on 'double summation'.**

The results stated below on 'double summation' are left as exercises. These results can be generalized to 'triple summation' et cetera.

Theorem (2).

Let m, n, p, q be integers. Suppose $m \leq n$ and $p \leq q$. Suppose

$$\begin{matrix} a_{m,p}, & a_{m,p+1}, & \cdots & a_{m,q}, \\ a_{m+1,p}, & a_{m+1,p+1}, & \cdots & a_{m+1,q}, \\ \vdots & \vdots & & \vdots \\ a_{n,p}, & a_{n,p+1}, & \cdots & a_{n,q} \end{matrix}$$

are numbers. Then $\displaystyle\sum_{j=m}^{n}\sum_{k=p}^{q} a_{j,k} = \sum_{k=p}^{q}\sum_{j=m}^{n} a_{j,k}$.

Corollary (3).

Let m, n be integers. Suppose $m \leq n$. Suppose

$$\begin{matrix} a_{m,m}, & a_{m,m+1}, & a_{m,m+2} & \cdots & a_{m,n-1} & a_{m,n}, \\ & a_{m+1,m+1}, & a_{m+1,m+2} & \cdots & a_{m+1,n-1} & a_{m+1,n}, \\ & & a_{m+2,m+2} & \cdots & a_{m+2,n-1} & a_{m+2,n}, \\ & & & \ddots & \vdots & \vdots \\ & & & & a_{n-1,n-1}, & a_{n-1,n}, \\ & & & & & a_{n,n} \end{matrix}$$

are numbers. Then $\displaystyle\sum_{j=m}^{n}\sum_{k=j+1}^{n} a_{j,k} = \sum_{k=m}^{n}\sum_{j=m}^{k} a_{j,k}$.

Theorem (4).

Let m, n, p, q be integers. Suppose $m \leq n$ and $p \leq q$.

Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n$, $b_p, b_{p+1}, b_{p+2}, \cdots, b_{q-1}, b_q$ are numbers.

Then $\displaystyle\left(\sum_{j=m}^{n} a_j\right)\left(\sum_{k=p}^{q} b_k\right) = \sum_{j=m}^{n}\sum_{k=p}^{q} a_j b_k = \sum_{k=p}^{q}\sum_{j=m}^{n} a_j b_k$.

5. **Convention on the product symbol.**

Suppose m, n are integers.

(a) *Suppose $m \leq n$. Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n$ are numbers.*

We agree to write $a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_{n-1} \cdot a_n$ as $\displaystyle\prod_{k=m}^{n} a_k$.

(b) *Suppose $m > n$. Then we agree to read* $\displaystyle\prod_{k=m}^{n} b_k$ *as 1 no matter what the b_k's are.*

Remark. The symbol k in the expression $\displaystyle\prod_{k=m}^{n} a_k$ is called the dummy index in this expression.

6. **Theorem (5). (Basic properties of the product symbol.)**

Let m, n, p, q be integers. Suppose $m \leq n \leq p$. Let $a_m, a_{m+1}, \cdots, a_n, a_{n+1}, \cdots, a_p, b_m, b_{m+1}, \cdots, b_n, b_{n+1}, \cdots, b_p, c$. The statements below hold:

(a) $\prod_{k=m}^{n} a_k = \prod_{j=m}^{n} a_j$.

(b) $\prod_{k=m}^{m} a_k = a_m$.

(c) $\prod_{k=m}^{n} a_k = \prod_{j=m-q}^{n-q} a_{j+q}$.

(d) $\prod_{k=m}^{n} a_k = \prod_{j=m}^{n} a_{m+n-j}$.

(e) $\prod_{k=m}^{n} c = c^{n-m+1}$.

(f) $\prod_{k=m}^{n} ca_k = c^{n-m+1} \prod_{k=m}^{n} a_k$.

(g) $\prod_{k=m}^{n} (a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$.

(h) $\prod_{k=m}^{p} a_k = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=n+1}^{p} a_k\right)$.

Remark. Do not be scared by the symbols. Many of these formulae are just 'short-hand' for what you are very familiar. For instance:

- '$\prod_{k=m}^{n} (a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$' is '$a_m b_m \cdot a_{m+1} b_{m+1} \cdot \ldots \cdot a_n b_n = (a_m \cdot a_{m+1} \cdot \ldots \cdot a_n)(b_m \cdot b_{m+1} \cdot \ldots \cdot b_n)$' in disguise.

Proof. Apply mathematical induction. (Hence they are postponed at this moment but will be left to you).

7. **Telescopic method for sums and products.**

Suppose we are given some 'finite sequence' of numbers $b_m, b_{m+1}, b_{m+2}, \cdots, b_n$, and we want to compute the sum $\sum_{k=m}^{n} b_k$. If it happens that there are some appropriate numbers $a_m, a_{m+1}, a_{m+2}, \cdots, a_n, a_{n+1}$ for which $b_j = a_{j+1} - a_j$ holds for each j, then we obtain the equality

$$\sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$$

immediately. This method for computing/simplifying sums is referred to as the **telescopic method for sums**.

There is an analogous **telescopic method for products**; formulate it as an exercise.

8. **Theorem (6). (Mechanism behind the Telescopic Method.)**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers.

(a) *Whenever $m \leq n$, the equality $\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$ holds.*

(b) *Further suppose $a_j \neq 0$ for each j. Then whenever $m \leq n$, the equality $\prod_{k=m}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_m}$ holds.*

Proof.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers.

(a) Let m be an integer.

For any integer n no less than m, denote by $P(n)$ the proposition $\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$.

- We have $\sum_{k=m}^{m} (a_{k+1} - a_k) = a_{m+1} - a_m$.
  Then $P(m)$ is true.
- Let ℓ be an integer no less than m. Suppose $P(\ell)$ is true.
  Then $\sum_{k=m}^{\ell} (a_{k+1} - a_k) = a_{\ell+1} - a_m$
  We deduce that $P(\ell + 1)$ is true:

We have $\sum_{k=m}^{\ell+1}(a_{k+1}-a_k) = \sum_{k=m}^{\ell}(a_{k+1}-a_k) + \sum_{k=\ell+1}^{\ell+1}(a_{k+1}-a_k) = (a_\ell - a_m) + (a_{\ell+1}-a_\ell) = a_{\ell+1}-a_m$.
Then $P(\ell+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any integer n no less than m.

(b) Exercise.

9. **Illustrations on the Telescopic Method.**

(a) Note that $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$ any number x equal to neither 0 nor -1.

Then

$$\sum_{j=1}^{n}\frac{1}{j(j+1)} = \sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+1}\right) = 1-\frac{1}{n+1}$$

(b) Note that $4(x+1)(x+2)(x+3) = (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3)$ any number x.
Then

$$\begin{aligned}\sum_{j=0}^{n}4(j+1)(j+2)(j+3) &= \sum_{j=0}^{n}[(j+1)(j+2)(j+3)(j+4)-j(j+1)(j+2)(j+3)] \\ &= (n+1)(n+2)(n+3)(n+4) - 0\cdot(0+1)(0+2)(0+3) \\ &= (n+1)(n+2)(n+3)(n+4)\end{aligned}$$

Therefore $\sum_{j=0}^{n}4(j+1)(j+2)(j+3) = \dfrac{1}{4}(n+1)(n+3)(n+3)(n+4)$.