MATH1050 Quadratic polynomials

0. *You are supposed to have a lot of practical experience in the handling of* **polynomial expressions**, **linear/quadratic functions**, **linear/quadratic equations** *in school mathematics. You are supposed to be familiar with the* **notion of real-valued functions of one-real variable** *in school mathematics.*

1. **Definition. (Linear and quadratic polynomials.)**

   Let a, b, c be numbers. Consider the formal sum $ax^2 + bx + c$.

   (a) *Suppose $a \neq 0$. Then this formal sum is called a quadratic polynomial with indeterminate x.*

   (b) *Suppose $a = 0$ and $b \neq 0$. Then this formal sum is called a linear polynomial with indeterminate x.*

   (c) *Suppose $a = b = 0$. Then this formal sum is called a constant polynomial.*

   Remark. Such a formal sum $ax^2 + bx + c$ is a special instance of some more general objects known as polynomials with one indeterminate.

   Terminology and convention.

   - **Notations.** For convenience, we agree to use the functional notation (such as $f(x)$, $g(x)$, $h(x)$) for denoting such polynomials.
   - **Equality for polynomials.** Suppose $f(x) = ax^2 + bx + c$, $g(x) = a'x^2 + b'x + c'$ are polynomials. We agree to declare that $f(x)$ is the same as $g(x)$ exactly when $a = a'$, $b = b'$ and $c = c'$. In this situation We write $f(x) = g(x)$ as polynomials.

   Further remark. We take for granted what we have been told at school about addition, subtraction and multiplication for polynomials:

   ∗ $(ax^2 + bx + c) + (a'x^2 + b'x + c') = (a + a')x^2 + (b + b')x + (c + c')$ whenever a, b, c, a', b', c' are real numbers.

   ∗ $A(ax^2 + bx + c) = (Aa)x^2 + (Ab)x + (Ac)$ whenever a, b, c, A are real numbers.

   ∗ $(bx + c)(b'x + c') = (bb')x^2 + (bc' + b'c)x + cc'$ whenever b, c, b', c' are real numbers.

   Note that each of the equalities above is an equality for polynomials; they should be understood in the sense that the respective coefficients in the polynomials of the two sides of the symbol '=' agree with each other. (In school maths textbooks, the words *identical as polynomials*, *polynomial identities* are used instead of *equal as polynomial*, *polynomial equalities* here.)

2. **Definition. (Roots of linear and quadratic polynomials.)**

   Let $f(x)$ be a linear or quadratic polynomial, given by $f(x) = ax^2 + bx + c$. Let α be a number.

   Suppose that upon the substitution of the indeterminate x in $f(x)$ by '$x = \alpha$', we obtain the equality (of numbers) $a\alpha^2 + b\alpha + c = 0$. Then we say α is a root of $f(x)$.

   Terminology and convention.

   - **Notations.** For convenience, we agree to write $f(\alpha) = 0$ here exactly when α is a root of $f(x)$.
   - *If α is a real number, we say that α is a root of $f(x)$ in $\mathbb{R}$. If α is a complex number, we say that α is a root of $f(x)$ in $\mathbb{C}$. In general, if α is a number amongst a specific collection of numbers, say, so-and-so, then we say that α is a root of $f(x)$ in so-and-so.*

3. **Theorem (1). (Roots of quadratic polynomials with real coefficients.)**

Let a, b, c be real numbers, with $a \neq 0$. Let α be a number. Let $f(x)$ be the quadratic polynomial given by $f(x) = ax^2 + bx + c$.

(a) *Suppose α is a root of $f(x)$. Let $\beta = -\dfrac{b}{a} - \alpha$. Then the statements below hold:*

   i. *$f(x) = a(x-\alpha)(x-\beta)$ as polynomials.*

   ii. *β is a root of $f(x)$.*

   iii. *$\alpha\beta = \dfrac{c}{a}$.*

(b) *Define $\Delta_f = b^2 - 4ac$. We call Δ_f the discriminant of the polynomial $f(x)$. Then the statements below hold:*

   i. *$f(x) = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right]$ as polynomials.*
   (This polynomial equality is referred as 'completing the square for the quadratic polynomial $f(x)$'.)

   ii. *Suppose $\Delta_f \geq 0$. Define $\alpha_\pm = \dfrac{-b \pm \sqrt{\Delta_f}}{2a}$ respectively. Then $f(x) = a(x-\alpha_+)(x-\alpha_-)$ as polynomials.*

   iii. *Now suppose $\Delta_f < 0$ instead. Define $\zeta = \dfrac{-b + i\sqrt{-\Delta_f}}{2a}$. Further define $\bar{\zeta} = \dfrac{-b - i\sqrt{-\Delta_f}}{2a}$. Then $f(x) = a(x-\zeta)(x-\bar{\zeta})$ as polynomials.*

Proof. Exercise in school maths.

Remark. What Theorem (1) says is that each quadratic polynomial with real coefficients $f(x)$ has a pair of roots and 'factorizes into linear polynomials'. Moreover, if the polynomial $f(x)$ is given by $f(x) = ax^2 + bx + c$ and the pair of roots concerned are α, β, then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$. Furthermore, regarding the quadratic equation

$$ax^2 + bx + c = 0 \quad \text{——} \quad (\star)$$

with unknown x, there are exactly three mutually exclusive possibilities:

(1) Suppose $\Delta_f > 0$. Then the equation $(\star)$ has exactly two distinct solutions amongst the real numbers.

(2) Suppose $\Delta_f = 0$. Then the equation $(\star)$ has exactly one repeated solution amongst the real numbers.

(3) Suppose $\Delta_f < 0$. Then the equation $(\star)$ has exactly two solutions, themselves complex conjugates of each other, amongst the complex numbers (but outside the reals).

In any case, the equation $(\star)$ has at least one solution amongst the complex numbers.

4. **Real-valued functions of one real variable defined by linear/quadratic polynomials.**

Definition. (Affine linear functions.)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (real-valued) function (of one real variable). Then f is said to be a(n) (affine) linear function if there exist some $b, c \in \mathbb{R}$ such that $f(x) = bx + c$ for any $x \in \mathbb{R}$.

Remark. Hence the 'formula of definition' of such a function f is given by a linear polynomial with real coefficients.

Further remark on coordinate geometry. The graph $y = f(x)$ of such a linear function f is given by the 'infinite straight line' $y = bx + c$.

Definition. (Quadratic functions.)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (real-valued) function (of one real variable). Then f is said to be a quadratic function if there exist some $a, b, c \in \mathbb{R}$ such that $a \neq 0$ and $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

Remark. Hence the 'formula of definition' of such a function f is given by a quadratic polynomial with real coefficients.

Further remark on coordinate geometry. The graph $y = f(x)$ of such a quadratic function f is a curve known as the parabola (on the coordinate plane). The point $(-\dfrac{b}{2a}, -\dfrac{\Delta_f}{4a})$, where the quadratic function f attains absolute extrema and where f 'changes from' being strictly increasing/decreasing to strictly decreasing/increasing, is known as the vertex of the parabola $y = f(x)$.

5. **Strict monotonicity for quadratic functions with real coefficients of one real variable.**

Definition. (Strict monotonicity.)

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a function with domain D which contains I entirely.

(a) *h is said to be strictly increasing on I if for any $s, t \in I$, the inequality $h(s) < h(t)$ holds.*

(b) *h is said to be strictly decreasing on I if for any $s, t \in I$, the inequality $h(s) > h(t)$ holds.*

Theorem (2). (Strict monotonicity for quadratic functions.)

Let $a, b, c \in \mathbb{R}$, with $a \neq 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

(a) *Suppose $a > 0$. Then f is strictly decreasing on $(-\infty, -\frac{b}{2a}]$ and strictly increasing on $[-\frac{b}{2a}, +\infty)$.*

(b) *Suppose $a < 0$. Then f is strictly increasing on $(-\infty, -\frac{b}{2a}]$ and strictly decreasing on $[-\frac{b}{2a}, +\infty)$.*

Proof. Exercise. (Outline of argument. Start by factorizing the expression $f(s) - f(t)$, extracting the factor $s - t$. Then ask what may be said of the number $f(s) - f(t)$ under the assumption $s < t < -b/(2a)$. Et cetera.)

Remark on geometric interpretation. Suppose $a > 0$. Then the curve $y = f(x)$ will 'drop and drop' as the value of x increases from the 'negative infinity' to $-\frac{b}{2a}$, and will 'rise and rise' as the value of x increases from $-\frac{b}{2a}$ to the 'positive infinity'.

6. **Absolute extrema for quadratic functions with real coefficients of one real variable.**

Definition. (Absolute extrema.)

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a function with domain D which contains I entirely. Let p be a point in I.

(a) *h is said to attain absolute maximum at p on I if for any $x \in I$, the inequality $h(x) \leq h(p)$ holds. The number $h(p)$ is called the absolute maximum value of h on I.*

(b) *h is said to attain absolute minimum at p on I if for any $x \in I$, the inequality $h(x) \geq h(p)$ holds. The number $h(p)$ is called the absolute minimum value of h on I.*

Theorem (3). (Absolute extrema for quadratic functions.)

Let $a, b, c \in \mathbb{R}$, with $a \neq 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$. Denote the discriminant of $f(x)$ by Δ_f.

(a) *Suppose $a > 0$. Then f attains absolute minimum at $-\frac{b}{2a}$ on $\mathbb{R}$, with absolute minimum value $-\frac{\Delta_f}{4a}$.*

(b) *Suppose $a < 0$. Then f attains absolute maximum at $-\frac{b}{2a}$ on $\mathbb{R}$, with absolute maximum value $-\frac{\Delta_f}{4a}$.*

Proof. Exercise. (The key of the argument is in making use of 'completing the square'.)

Remark on geometric interpretation. Suppose $a > 0$. Then the curve $y = f(x)$ will 'touch the bottom' as the value of x, varying amongst all positive real numbers, reaches $-\frac{b}{2a}$.

Corollary to Theorem (3).

Let $a, b, c \in \mathbb{R}$. Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

Then the statements (†), (‡) are logically equivalent:

(†) *$f(x) \geq 0$ for any $x \in \mathbb{R}$.*

(‡) $\Delta_f \leq 0$.

Equality in (‡) holds iff $-\frac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

Remark. This result will play a key role in the proof of the Cauchy-Schwarz Inequality.

Proof of Corollary to Theorem (3).

Let $a, b, c \in \mathbb{R}$. Suppose $a > 0$, $\Delta_f = b^2 - 4ac$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the quadratic polynomial function defined by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

By Theorem (3), f attains the absolute minimum at $-\dfrac{b}{2a}$, with $f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.

- $[(\dagger) \Longrightarrow (\ddagger)?]$
  Suppose $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  Note that $-\dfrac{b}{2a} \in \mathbb{R}$. Then, by assumption, we have $0 \leq f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a}$.
  Since $a > 0$, we have $-4a < 0$. Then $\Delta_f = -4a \cdot \left(-\dfrac{\Delta_f}{4a}\right) \leq 0$.
- $[(\ddagger) \Longrightarrow (\dagger)?]$
  Suppose $\Delta \leq 0$. Then, since $a > 0$, we have $-\dfrac{\Delta_f}{4a} \geq 0$.
  Pick any $x \in \mathbb{R}$. We have $f(x) \geq f(-\dfrac{b}{2a}) = -\dfrac{\Delta_f}{4a} \geq 0$.

By Theorem (1), $\Delta_f = 0$ iff $-\dfrac{b}{2a}$ is a repeated real root of the polynomial $f(x)$.

7. **Strict convexity/concavity for quadratic functions with real coefficients of one real variable.**

Definition. (Strict convexity/concavity.)

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a function with domain D which contains I entirely.

(a) *h is said to be strictly convex on I if for any $p, q \in I$, for any $\lambda \in (0, 1)$ the inequality $h((1 - \lambda)p + \lambda q) < (1 - \lambda)h(p) + \lambda h(q)$ holds.*

(b) *h is said to be strictly concave on I if for any $p, q \in I$, for any $\lambda \in (0, 1)$ the inequality $h((1 - \lambda)p + \lambda q) > (1 - \lambda)h(p) + \lambda h(q)$ holds.*

Theorem (4). (Strict convexity/concavity of quadratic functions.)

Let $a, b, c \in \mathbb{R}$, with $a \neq 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the quadratic function given by $f(x) = ax^2 + bx + c$ for any $x \in \mathbb{R}$.

(a) *Suppose $a > 0$. Then f is strictly convex on $\mathbb{R}$.*

(b) *Suppose $a < 0$. Then f is strictly concave on $\mathbb{R}$.*

Proof. Exercise. (Nothing but a tedious computation.)

Remark on geometric interpretation. Suppose $a > 0$. Join any two points on the curve $y = f(x)$ by a line segment, and it will happen that every point on the curve 'between' these two points will be 'below' the the point with the same x-coordinate in the line segment. An equivalent description is that a variable point on the curve $y = f(x)$ 'moving' from the 'negative infinity' to the 'positive infinity' will be 'turning left' all the way.

8. **Appendix. Beyond school maths: complex-valued linear/quadratic functions of one complex variable.**

Suppose a, b, c are complex numbers, and $f(z)$ the polynomial with indeterminate z given by $f(z) = az^2 + bz + c$. For each complex number α, upon the substitution of the indeterminate z in $f(z)$ by '$z = \alpha$', we obtain the complex number $a\alpha^2 + b\alpha + c$ (which is uniquely determined by the value of α). This way of assigning complex numbers to complex numbers defines a 'complex-valued function of one complex variable' whose domain is $\mathbb{C}$ and whose range is also $\mathbb{C}$. For convenience, we also denote such a function by the symbol f (with which we label the polynomial $az^2 + bz + c$). When $a = 0$, we refer to this function as a(n) (affine) linear function from $\mathbb{C}$ to $\mathbb{C}$. When $a \neq 0$, we refer to this function as a quadratic function from $\mathbb{C}$ to $\mathbb{C}$. Such a function is a simple example of functions beyond what we saw in school maths.