MATH1050 Proof-writing Exercise 9 (Answers and selected solutions)

1. (a) **Solution.**

    i. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
    Suppose $g \circ f$ is surjective.
    Pick any $z \in C$.
    Since $g \circ f$ is surjective, there exists some $x \in A$ such that $z = (g \circ f)(x)$.
    Define $y = f(x)$. We have $y \in B$.
    For the same $x \in A$, $y \in B$ and $z \in C$, we have $g(y) = g(f(x)) = (g \circ f)(x) = z$.
    It follows that g is surjective.

    ii. Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
    Suppose $g \circ f$ is injective.
    Pick any $x, w \in A$. Suppose $f(x) = f(w)$.
    Then $g(f(x)) = g(f(w))$.
    Note that $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(w) = g(f(w))$.
    Then $(g \circ f)(x) = (g \circ f)(w)$.
    Since $g \circ f$ is injective, we have $x = w$.
    It follows that f is injective.

   (b) ——

2. **Solution.**

   (a) Take $A = \{0\}$, $B = \{1, 2\}$, $C = \{3\}$. Here $0, 1, 2, 3$ are pairwise distinct objects.
   Define the function $f : A \longrightarrow B$ by $f(0) = 1$. Define the function $g : B \longrightarrow C$ by $g(1) = g(2) = 3$.
   The function $g \circ f : A \longrightarrow C$ is given by $(g \circ f)(0) = g(f(0)) = g(1) = 3$.
   Pick any $z \in C$. Since $C = \{3\}$, we have $z = 3$. Note that $0 \in A$ and $(g \circ f)(0) = 3 = z$. It follows that $g \circ f$ is surjective.
   Note that $2 \in B$. Pick any $x \in A$. Since $A = \{0\}$, we have $x = 0$. Then $f(x) = f(0) = 1 \neq 2$. Therefore the function f is not surjective.

   (b) Take $A = \{0\}$, $B = \{1, 2\}$, $C = \{3\}$. Here $0, 1, 2, 3$ are pairwise distinct objects. Define the function $f : A \longrightarrow B$ by $f(0) = 1$. Define the function $g : B \longrightarrow C$ by $g(1) = g(2) = 3$.
   The function $g \circ f : A \longrightarrow C$ is given by $(g \circ f)(0) = g(f(0)) = g(1) = 3$.
   Pick any $x, w \in A$. Suppose $(g \circ f)(x) = (g \circ f)(w)$. Since $A = \{0\}$, we have $x = 0 = w$. It follows that $g \circ f$ is injective.
   Note that $1, 2 \in B$, $1 \neq 2$ and $g(1) = g(2) = 3$. Then the function g is not injective.

3. **Answer.**

   (a) True.

   (b) False.

4. **Answer.**

   The statement ($\sharp$) is true.

5. **Solution.**

   (a) The statement ($\sharp$) is true. We give a proof:

    Let A, B be sets, and $f : A \longrightarrow B$ be a function.
    Let U, V be subsets of B. Suppose $U \subset V$.
    Pick any object x. Suppose $x \in f^{-1}(U)$.
    By the definition of pre-image sets, there exists some $y \in U$ such that $y = f(x)$.
    Since $y \in U$ and $U \subset V$, we have $y \in V$.
    Since $y \in V$ and $y = f(x)$, we have $x \in f^{-1}(V)$ according to the definition of pre-image sets.
    It follows that $f^{-1}(U) \subset f^{-1}(V)$.

Acceptable answer.

Let A, B be sets, and $f : A \longrightarrow B$ be a function.
Let U, V be subsets of B. Suppose $U \subset V$.
Pick any object x. Suppose $x \in f^{-1}(U)$.
By the definition of pre-image sets, $f(x) \in U$.
Since $f(x) \in U$ and $U \subset V$, we have $f(x) \in V$.
Since $f(x) \in V$, we have $x \in f^{-1}(V)$ according to the definition of pre-image sets.
It follows that $f^{-1}(U) \subset f^{-1}(V)$.

(b) The statement ($\flat$) is false. We give a dis-proof by counter-example.

Take $A = \{0, 1\}$, $B = \{0, 1\}$. Here $0, 1$ are distinct objects.
Define the function $f : A \longrightarrow B$ by $f(0) = f(1) = 0$.
Take $U = \{0, 1\}$, $V = \{0\}$.
We have $f^{-1}(U) = f^{-1}(V) = \{0, 1\}$. Then $f^{-1}(U) \subset f^{-1}(V)$.
Note that $1 \in U$ and $1 \notin V$. Then $U \not\subset V$.

6. **Answer.**

(a) The statement ($\sharp$) is true.

(b) The statement ($\flat$) is false.