MATH1050 Proof-writing Exercise 9

Advice.

- All the questions are concerned with proofs and dis-proofs on abstract and theoretical statements about functions.

  Study the Handouts *Surjectivity and Injectivity*, *Compositions*, *Surjectivity and Injectivity*, *Image sets and pre-image sets*, *Theoretical results involving image sets and pre-image sets*, *Characterization of surjectivity with image sets, pre-image sets.*

- Besides the handouts mentioned above, Questions (5), (8), of Assignment 9 and Question (9) of Assignment 10 may also be relevant.

1. (a) Prove each of the statements below:
   i. *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then g is surjective.*
   ii. *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then f is injective.*

   (b)♣ Let I, J, K be sets, and $\alpha : I \longrightarrow J$, $\beta : J \longrightarrow K$, $\gamma : K \longrightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha$, $\alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.
   Prove that each of the functions α, β, γ is both surjective and injective.

2. Dis-prove each of the statements below by giving an appropriate argument.
   (a) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then f is surjective.*
   (b) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then g is injective.*

3.◇ We introduce this definition for the notion of *union of functions* below:

   - *Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$.*

     Define the function $f \cup g : A \cup B \longrightarrow C \cup D$ by $(f \cup g)(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ g(x) & \text{if} \quad x \in B \end{cases}$.

     The function $f \cup g$ is called the **union of the functions** *f, g.*

   Consider each of the statements below. Determine whether it is true or false. Justify your answer with an appropriate argument.

   (a) *Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are surjective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is surjective.*
   (b) *Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are injective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is injective.*

4.♣ Determine whether the statement (♯) is true. Justify your answer with an appropriate argument:

   (♯) *Let A, B be sets, and $f : A \longrightarrow B, g : A \longrightarrow B$ be functions. Suppose $\Phi : \mathsf{Map}(B, A) \longrightarrow \mathsf{Map}(A, B)$ is the function defined by $\Phi(p) = g \circ p \circ f$ for any $p \in \mathsf{Map}(B, A)$. Further suppose f is surjective and g is injective. Then Φ is injective.*

5. (a)◇ Is the statement (♯) true? Justify your answer with reference to the definition of *pre-image sets*:
   (♯) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B. Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.*

   (b)◇ Is the statement (♭) true? Justify your answer with reference to the definition of *pre-image sets*:
   (♭) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B. Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.*

6. (a)◇ Is the statement (♯) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:
   (♯) *Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then for any subset S of A, $S \subset f^{-1}(f(S))$.*

   (b)◇ Is the statement (♭) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:
   (♭) *Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then for any subset S of A, $f^{-1}(f(S)) \subset S$.*