MATH1050 Proof-writing Exercise 8 (Answers and selected solutions)

1. **Solution.**

   (a) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2\bar{z}$ for any $z \in \mathbb{C}$.

      i. [We want to verify the statement '*for any* $w \in \mathbb{C}$, *there exists some* $z \in \mathbb{C}$ *such that* $f(z) = w$.']
      Pick any $w \in \mathbb{C}$. We have $w = 0$ or $w \neq 0$.
      * (Case 1.) Suppose $w = 0$. Then $f(0) = 0 = w$.
      * (Case 2.) Suppose $w \neq 0$. (Note that $|w| \neq 0$. Then $\dfrac{1}{\sqrt[3]{|w|}}$ is well-defined as a complex number.)
        Define $z = \dfrac{w}{(\sqrt[3]{|w|})^2}$. By definition, $z \in \mathbb{C}$.
        We have $f(z) = z^2\bar{z} = \left(\dfrac{w}{(\sqrt[3]{|w|})^2}\right)^2 \overline{\left[\dfrac{w}{(\sqrt[3]{|w|})^2}\right]} = \dfrac{w^2\bar{w}}{(\sqrt[3]{|w|})^6} = \dfrac{w|w|^2}{|w|^2} = w$.

      It follows that f is surjective.

      ii. [We want to verify the statement '*for any* $u, v \in \mathbb{C}$, *if* $f(u) = f(v)$ *then* $u = v$.']
      Pick any $u, v \in \mathbb{C}$. Suppose $f(u) = f(v)$.
      Then $|u|^3 = |u^2\bar{u}| = |f(u)| = |f(v)| = |v^2\bar{v}| = |v|^3$.
      Since $|u|, |v| \in \mathbb{R}$, we have $|u| = |v|$.
      Now $u|u|^2 = u^2\bar{u} = f(u) = f(v) = v^2\bar{v} = v|v|^2 = v|u|^2$.
      Then $(u - v)|u|^2 = 0$. Therefore $u - v = 0$ or $|u|^2 = 0$.
      * (Csae 1.) Suppose $u - v = 0$. Then $u = v$.
      * (Case 2.) Suppose $u - v \neq 0$. Then $|u|^2 = 0$. Therefore $u = 0$.
        Also, $|v| = |u| = 0$. Then $u = 0 = v$.

      Hence, in any case $u = v$.
      It follows that f is injective.

   (b) Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Let $g : \mathbb{C}^* \longrightarrow \mathbb{C}$ be the function defined by $g(z) = z/\bar{z}$ for any $z \in \mathbb{C}^*$.

      i. [We want to verify the statement '*there exist some* $u_0, v_0 \in \mathbb{C}^*$ *such that* $g(u_0) = g(v_0)$ *and* $u_0 \neq v_0$.']
      Take $u_0 = 1$, $v_0 = -1$.
      We have $u_0, v_0 \in \mathbb{C}^*$ and $u_0 \neq v_0$.
      We have $g(u_0) = 1$ and $g(v_0) = 1$. Then $g(u_0) = g(v_0)$.
      It follows that g is not injective.

      ii. [We want to verify the statement '*There exists some* $w_0 \in \mathbb{C}$ *such that for any* $z \in \mathbb{C}^*$, $g(z) \neq w_0$.']
      Take $w_0 = 2$. Note that $w_0 \in \mathbb{C}$. We verify with the proof-by-contradiction method, that for any $z \in \mathbb{C}^*$, $g(z) \neq w_0$:
      - Suppose there existed some $z \in \mathbb{C}^*$ such that $g(z) = w_0$.
        Then $|g(z)| = |w_0| = 2$.
        For the same z, we have $|g(z)| = |z/\bar{z}| = |z|/|\bar{z}| = |z|/|z| = 1 \neq 2$.
        Now $|g(z)| = 2$ and $|g(z)| \neq 2$. Contradiction arises.

      It follows that g is not surjective.

2. **Solution.**

   Denote the interval $(0, +\infty)$ by I. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ for any $x \in I$.

   (a) Pick any $x, w \in I$. Suppose $f(x) = f(w)$. Then $\dfrac{1}{2}\left(x - \dfrac{1}{x}\right) = \dfrac{1}{2}\left(w - \dfrac{1}{w}\right)$. We have $x^2w - w = w^2x - x$.
   Therefore $xw(x - w) = x^2w - w^2x = w - x$. Hence $(xw + 1)(x - w) = 0$.
   Since $x, w \in I$, we have $x > 0, w > 0$ and $xw + 1 > 0$. Therefore $x - w = 0$. Hence $x = w$.
   It follows that f is injective.

(b) Pick any $y \in \mathbb{R}$. Take $x = y + \sqrt{y^2+1}$. Note that $x \in I$.

We have $f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right) = \frac{1}{2}\left(y + \sqrt{y^2+1} - \frac{1}{y+\sqrt{y^2+1}}\right)$

$= \frac{1}{2}\left[y + \sqrt{y^2+1} - \frac{y - \sqrt{y^2+1}}{(y+\sqrt{y^2+1})(y-\sqrt{y^2+1})}\right] = \frac{1}{2}(y + \sqrt{y^2+1} + y - \sqrt{y^2+1}) = y.$

It follows that f is surjective.

3. **Solution.**

Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\frac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

(a) Take $x_0 = \frac{2^{2/3}}{\pi^{2/3}}$, $w_0 = \frac{2^{2/3}}{3^{2/3}\pi^{2/3}}$. Note that $x_0, w_0 \in (0, +\infty)$ and $x_0 \neq w_0$. Also note that $f(x_0) = 0 = f(w_0)$. It follows that f is not injective.

(b) i. Let $x \in (0, +\infty)$. We have $|x^2 - 2x + 4| \leq |x^2 + 4| + |2x| = x^2 + 2x + 4 = |x^2 + 2x + 4|$. Note that $|x^2 + 2x + 4| > 0$. Then $\left|\frac{x^2 - 2x + 4}{x^2 + 2x + 4}\right| \leq 1$.

ii. Take $y_0 = 2$. We verify that for any $x \in (0, +\infty)$, we have $f(x) \neq y_0$:

- Let $x \in (0, +\infty)$. We have $|f(x)| = \left|\frac{x^2-2x+4}{x^2+2x+4} \cos\left(\frac{1}{x\sqrt{x}}\right)\right| \leq \left|\frac{x^2-2x+4}{x^2+2x+4}\right| \cdot \left|\cos\left(\frac{1}{x\sqrt{x}}\right)\right| \leq 1 \cdot 1 = 1 < 2$.
  Then $f(x) \neq 2$.

It follows that f is not surjective.

4. **Answer.**

(a) No. Note that $f(0) = f(1)$.

(b) No. Note that $f(x) \neq 1$ for any $x \in \mathbb{R}$.

5. **Answer.**

(a) ——

(b) No. Note that $f(1) = f(\cos(2\pi/n) + i\sin(2\pi/n))$.

6. **Answer.**

(a) Yes. You may need to use the Fundamental Theorem of Algebra here.

(b) No. Observe that $h(2) = h(-2)$.