MATH1050 Proof-writing Exercise 8

Advice.

- All the questions are concerned with concrete examples on surjectivity and injectivity.

  Study the Handouts *Surjectivity and Injectivity*, *Surjectivity and injectivity for 'nice' real-valued functions of one real variable*, *Surjectivity and injectivity for 'simple' complex-valued functions of one complex variable.*

- Besides the handouts mentioned above, Questions (2), (3), (4) of Assignment 9 may also be relevant.

1. *In this question, you will want to make good use of the basic properties of modulus of a complex number.*

   You are advised against using arguments of complex numbers, and also agaisnt 'decomposing' complex numbers into real and imaginary parts. In fact, there is no need to use such things.

   (a)$^\diamondsuit$ Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2\bar{z}$ for any $z \in \mathbb{C}$.

   i. Verify that f is surjective, with reference to the definition of surjectivity.

   ii. Verify that f is injective, with reference to the definition of injectivity.

   (b) Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Let $g : \mathbb{C}^* \longrightarrow \mathbb{C}$ be the function defined by $g(z) = z/\bar{z}$ for any $z \in \mathbb{C}^*$.

   i. Verify that g is not injective, with reference to the definition of injectivity.

   ii. Verify that g is not surjective, with reference to the definition of surjectivity.

2. *Do not use any result from calculus of one real variable.*

   Denote the interval $(0, +\infty)$ by I. Let $f : I \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ for any $x \in I$.

   (a) Verify that f is injective, with reference to the definition of injectivity.

   (b) Verify that f is surjective, with reference to the definition of surjectivity.

3. *Do not use any result from calculus of one real variable.*

   Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\dfrac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

   (a) Verify that f is not injective, with reference to the definition of injectivity.

   (b) i. Verify that $\left|\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}\right| \leq 1$ for any $x \in (0, +\infty)$.

   ii. Apply the previous part, or otherwise, to verify that f is not surjective, with reference to the definition of surjectivity.

4.$^\diamondsuit$ Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ x^3 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

   (a) Is f injective? Justify your answer with reference to the definition of injectivity.

   (b) Is f surjective? Justify your answer with reference to the definition of surjectivity.

5. Let $n \in \mathbb{N}\backslash\{0, 1\}$, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^n$ for any $z \in \mathbb{C}$.

   (a) Verify that f is surjective. Justify your answer with reference to the definition of surjectivity.

   (b) Is f injective? Justify your answer with reference to the definition of injectivity.

6. We introduce/recall the definition for the notion of *zero of a function* below:

   Let D be a subset of $\mathbb{C}$, and $f : D \longrightarrow \mathbb{C}$ be a function. Let $\zeta \in D$. We say ζ is a **zero of f in D** *if $f(\zeta) = 0$.*

   In this question you may take for granted the validity of the result known as the **Fundamental Theorem of Algebra (FTA)***:*

(FTA) *Suppose f is a non-constant polynomial function from $\mathbb{C}$ to $\mathbb{C}$. Then f has a zero in $\mathbb{C}$.*

Let $D = \mathbb{C} \backslash \{1, -1, i, -i\}$, and $h : D \longrightarrow \mathbb{C}$ be the function defined by $h(z) = \dfrac{z^5 - 4z^3 - 2}{z^4 - 1}$ for any $z \in D$.

(a) Is h surjective? Prove your answer, with reference to the definition of surjectivity.

(b) Is h injective? Prove your answer, with reference to the definition of injectivity.