MATH1050 Proof-writing Exercise 7 (Answers and selected solutions)

1. (a) **Solution.**

Method (A).

Denote by N the statement below:

N: There exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

The negation of N reads:

$\sim N$: For any $x \in \mathbb{R}$, $x^2 + 2x + 3 \geq 0$.

We verify $\sim N$:

- Pick any $x \in \mathbb{R}$.
  We have $x^2 + 2x + 3 = (x+1)^2 + 2$. —— $(\star)$
  Since $x \in \mathbb{R}$, we have $x + 1 \in \mathbb{R}$. Then $(x+1)^2 \geq 0$.
  Therefore by $(\star)$, we have $x^2 + 2x + 3 \geq 0 + 2 = 2 \geq 0$.

Method (B).

[Denote by N the statement below:
N: *There exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.* .
We dis-prove the statement N by obtaining a contradiction from it.]

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

Note that $x^2 + 2x + 3 = (x+1)^2 + 2$. —— $(\star)$

Since $x \in \mathbb{R}$, we would have $x + 1 \in \mathbb{R}$. Then $(x+1)^2 \geq 0$.

By $(\star)$, we would have $x^2 + 2x + 3 \geq 0 + 2 = 2 \geq 0$.

Then $0 \leq x^2 + 2x + 3 < 0$. Contradiction arises.

Hence, in the first place, it is false that there exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

(b) ——

(c) ——

(d) **Solution.**

Method (A).

Denote by N the statement below:

N: *There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).*

The negation of N reads:

$\sim N$: For any $t \in \mathbb{R}$, there exists some $s \in \mathbb{C}$ such that $|s| > t$.

We verify $\sim N$:

- Pick any $t \in \mathbb{R}$.
  Take $s = |t| + 1$. By definition, $s \in \mathbb{C}$.
  Note that s is a positive real number. Then $|s| = ||t| + 1| = |t| + 1 > |t| \geq t$.

Method (B).

[Denote by N the statement below:
N: There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).
We dis-prove the statement N by obtaining a contradiction from it.]

Suppose it were true that there existed some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).

For this real number t, the statement '*for any $s \in \mathbb{C}$, $|s| \leq t$*' would be true.

Note that $|t| + 1$ is a complex number.

Then $||t| + 1| \leq t$.

Since $|t| + 1$ is a non-negative real number, we have $||t| + 1| = |t| + 1$.

Then we have $|t| + 1 \leq t \leq |t|$. Therefore $1 \leq 0$.

Contradiction arises.

2. (a) **Solution.**

Method (A).

Denote by N the statement below:

N: There exists some $x \in \mathbb{R}$ such that $|x+1| > |x| + 1$.

The negation of N reads:

$\sim N$: For any $x \in \mathbb{R}$, $|x+1| \leq |x| + 1$.

We verify $\sim N$:

- Pick any $x \in \mathbb{R}$. We have $x < -1$ or $-1 \leq x \leq 0$ or $x > 0$.

  (Case 1). Suppose $x < -1$. Then $x + 1 < 0$ and $x < 0$. We have $|x+1| = -(x+1) = -x-1 = |x| - 1 \leq |x| + 1$.

  (Case 2). Suppose $-1 \leq x \leq 0$. Then $x + 1 \geq 0$ also. We have $|x+1| = x+1 \leq 0+1 = 1 \leq |x|+1$.

  (Case 3). Suppose $x > 0$. Then $x + 1 > 0$ also. We have $|x+1| = x+1 = |x|+1 \leq |x|+1$.

  Hence, in any case, we have $|x+1| \leq |x| + 1$.

Alternative argument with Method (A).

Denote by N the statement below:

N: There exists some $x \in \mathbb{R}$ such that $|x+1| > |x| + 1$.

The negation of N reads:

$\sim N$: For any $x \in \mathbb{R}$, $|x+1| \leq |x| + 1$.

We verify $\sim N$:

Pick any $x \in \mathbb{R}$. Suppose it were true that $|x+1| > |x| + 1$ for this x.

Note that $|x+1| > |x| + 1 \geq 1 > 0$.

Then $x^2 + 2x + 1 = (x+1)^2 = |x+1|^2 > (|x|+1)^2 = x^2 + 2|x| + 1$.

Therefore $x > |x| \geq x$.

Contradiction arises.

Hence $|x+1| \leq |x| + 1$.

Method (B).

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $|x+1| > |x| + 1$.

Note that $|x+1| > |x| + 1 \geq 1 > 0$.

Then $x^2 + 2x + 1 = (x+1)^2 = |x+1|^2 > (|x|+1)^2 = x^2 + 2|x| + 1$.

Then $x > |x| \geq x$.

Contradiction arises.

(b) *Hint.*

Be aware that for any $z \in \mathbb{C}$, $z + 3 - 4i = z + (3 - 4i)$. Also note that $|3 - 4i| = 5$.

(c) *Hint.*

Be aware that for any $x \in \mathbb{R}$, $x + 4 = 2(x+1) + [-(x-2)]$.

(d) ——

3. ——

4. *Hint.*

Start the argument in this way:

> Suppose there existed some $k \in \mathbb{N}\backslash\{0,1\}$ such that for any positive integer n, the number $k^{1/n}$ was an integer.
>
> Define the set S by $S = \{x \in \mathbb{N}\backslash\{0,1\} : x = k^{1/n} \text{ for some } n \in \mathbb{N}\backslash\{0\}\}$.

Show that S is a non-empty subset of $\mathbb{N}$. Then apply the Well-ordering Principle for Integers to obtain a least element of S, say, u.

Obtain a contradiction by showing that there is an element of S, say, v, which is strictly less than u.

5. **Answer.**

(a) The statement is true.

(b) The statement is false.