MATH1050 Proof-writing Exercise 7

Advice.

- Almost all the questions are concerned with wholesale refutations.
- Study the Handout *Dis-proofs by wholesale refutation* before answering the questions.

  Also, make sure you know what it means (and what it takes) to correctly obtain the negation of a statement (which may involve one or more several quantifiers). Refer to the Handouts *Basics of logic in mathematics*, *Universal quantifier and existential quantifier*, *Statements with several quantifiers* on this matter.
- Besides the handout mentioned above, Question (2) of Assignment 7 is also suggestive on what it takes to give a correct wholesale-refutation argument.
- Sometimes you may want to apply the method of proof-by-contradiction within one passage of a proof.

  In this situation, it may be good to start that passage with the words '*we want to verify blah-blah-blah with the method of proof-by-contradiction* '.

  Be reminded that the assumptions used in such a passage of argument (which you hope will lead to a desired contradiction within that passage) must be stated clearly at the beginning of the passage concerned.

1. Dis-prove the statements below:

   (a) *There exists some* $x \in \mathbb{R}$ *such that* $x^2 + 2x + 3 < 0$.

   (b) *There exist some* $x, y \in \mathbb{R}\backslash\{0\}$ *such that* $(x + y)^2 = x^2 + y^2$.

   (c)$^\diamondsuit$ *There exists some* $r \in \mathbb{R}$ *such that* $r < r^5 \leq r^3$.

   (d)$^\diamondsuit$ *There exists some* $t \in \mathbb{R}$ *such that (for any* $s \in \mathbb{C}$, $|s| \leq t$*).*

2. Dis-prove the statements below. (Various results known as the Triangle Inequality may be useful.)

   (a) *There exists some* $x \in \mathbb{R}$ *such that* $|x + 1| > |x| + 1$.

   (b)$^\diamondsuit$ *There exists some* $z \in \mathbb{C}$ *such that* $|z + 3 - 4i| > |z| + 5$.

   (c)$^\diamondsuit$ *There exists some* $x \in \mathbb{R}$ *such that* $|x + 4| > 2|x + 1| + |x - 2|$.

   (d)$^\diamondsuit$ *There exist some* $a, b, c, r, s, t \in \mathbb{R}$ *such that*

   $$\sqrt{(a-r)^2 + (b-s)^2 + (c-t)^2} > \sqrt{(a-1)^2 + (b-2)^2 + (c-3)^2} + \sqrt{(r-1)^2 + (s-2)^2 + (t-3)^2}.$$

3. (a) Dis-prove the statement ($\star$):

   ($\star$) *There exist some positive real numbers* x, y *such that* $(x + y)^2 \leq x^2 + y^2$.

   (b)$^\diamondsuit$ Hence, or otherwise, dis-prove the statement ($\star\star$):

   ($\star\star$) *There exist some positive real numbers* u, v *such that* $\sqrt{u} + \sqrt{v} \leq \sqrt{u + v}$.

4.$^\clubsuit$ Dis-prove the statement ($\star$):

   ($\star$) *There exists some* $k \in \mathbb{N}\backslash\{0, 1\}$ *such that for any positive integer* n*, the number* $k^{1/n}$ *is an integer.*

   Remark. You will probably need the Well-ordering Principle for Integers.