MATH1050 Proof-writing Exercise 6 (Answers and selected solutions)

1. (a) **Answer.**
   There exist some $x, y, z \in \mathbb{N}$ such that $x + y$, $y + z$ are divisible by 3 and$x + z$ is not divisible by 3.
   (b) **Solution.**
   Take $x = z = 1$, $y = 2$.
   We have $x, y, z \in \mathbb{N}$.
   Note that $x + y = y + z = 3 = 1 \cdot 3$. We have $1 \in \mathbb{Z}$.
   Then, by definition, $x + y$, $y + z$ are divisible by 3.
   Note that $x + z = 2$. We verify that 2 is not divisible by 3:
   Suppose 2 were divisible by 3.
   Then there would exist some $k \in \mathbb{Z}$ such that $2 = 3k$.
   For the same k, we would have $k = \dfrac{2}{3}$. Then k is not an integer.
   Contradiction arises.

2. **Answer.**
   (a) One possible counter-example is given by: $x = y = 10$, and $z = 5$.
   (b) One possible counter-example is given by: $x = 1$, $y = 2$.
   (c) One possible counter-example is given by: $s = \sqrt{2}$, $t = -\sqrt{2}$.
   (d) One possible counter-example is given by: $a = 8$, $b = 9$, $c = 6$.
   (e) One possible counter-example is given by: $n = 3$, $\zeta = \cos\left(\dfrac{2\pi}{9}\right) + i\sin\left(\dfrac{2\pi}{9}\right)$.
   (f) One possible counter-example is given by: $n = 3$, $\zeta = \cos\left(\dfrac{2\pi}{9}\right) + i\sin\left(\dfrac{2\pi}{9}\right)$.

3. (a) **Answer.**
   There exist some sets A, B, C such that $A \backslash (C \backslash B) \not\subset A \cap B$.
   (b) **Solution.**
   Regard $0, 1, 2$ as distinct objects.
   Let $A = \{0, 1\}$, $B = \{1\}$, $C = \{2\}$.
   We have $A \cap B = B = \{1\}$, $C \backslash B = C = \{2\}$, $A \backslash (C \backslash B) = A = \{0, 1\}$.
   Note that $0 \in A \backslash (C \backslash B)$ and $0 \notin A \cap B$.
   Hence $A \backslash (C \backslash B) \not\subset A \cap B$.

4. **Answer.**
   (a) One possible counter-example is given by: $A = \{0\}$, $B = \{1\}$, $C = \{2\}$.
   (b) One possible counter-example is given by: $A = \{0\}, B = \{1\}, C = \{2\}$.
   (c) One possible counter-example is given by: $A = \{0\}$ and $B = C = \{0, 1\}$.
   (d) One possible counter-example is given by: $A = \{1\}, B = \{2\}, C = \{0, 1, 2\}$.
   (e) One possible counter-example is given by: $A = \{0, 2\}$, $B = \{1\}$, $C = \{1, 2\}$.
   (f) One possible counter-example is given by: $A = \{1\}$, $B = \{2\}$, $C = \{0, 1\}$, $D = \{0, 2\}$.

5. **Answer.**
   (a) One possible counter-example is given by: $I = \mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ given by $f(x) = x^3$ for any $x \in I$.
   (b) One possible counter-example is given by: $I = \mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ given by $f(x) = 0$ for any $x \in \mathbb{R}$.
   (c) One possible counter-example is given by: $I = (0, 1)$, $J = (1, 2)$,
   $f : I \cup J \longrightarrow \mathbb{R}$ given by $f(x) = \begin{cases} 0 & \text{if} \quad x \in I \\ 1 & \text{if} \quad x \in J \end{cases}$

6. **Answer.**

(a) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = A$.

(c) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.