MATH1050 Proof-writing Exercise 6

Advice.

- All the questions are concerned with dis-proofs-by-counter-examples.
- Study the Handout *Dis-proofs by counter-example.* before answering the questions.

  Also, make sure you know what it means (and what it takes) to correctly obtain the negation of a statement (which may involve one or more several quantifiers). Refer to the Handouts *Basics of logic in mathematics*, *Universal quantifier and existential quantifier*, *Statements with several quantifiers* on this matter.
- Besides the handout mentioned above, Question (2) of Assignment 6 is also suggestive on what it takes to give a correct dis-proof-by-counter-example argument.

1. Consider the statement ($\star$) below.

   ($\star$) *Let $x, y, z \in \mathbb{N}$. Suppose $x+y$, $y+z$ are divisible by 3. Then $x+z$ is divisible by 3.*

   (a) Write down the negation of the statement ($\star$).

   (b) Dis-prove the statement ($\star$).

2. Dis-prove each of the statements below. (It may help if you first find what the negation of the statement is.)

   (a) *Let $x, y, z \in \mathbb{N}$. Suppose $x-y>0$ and $x-z>0$ and $x-z$, $y-z$ are divisible by 5. Then $x+y+z$ is not divisible by 5.*

   (b) *Suppose $x, y \in \mathbb{N}$. Then $\sqrt{x^2+y^2} \in \mathbb{N}$.*

   (c)$^\diamond$ *For any $s, t \in \mathbb{R}$, if both of $s+t, st$ are rational, then at least one of s, t is rational.*

   (d)$^\diamond$ *For any $a, b, c \in \mathbb{N}$, if ab is divisible by c and $c<a$ and $c<b$, then at least one of a, b is divisible by c.*

   (e) *Let n be a positive integer, and ζ be a complex number. Suppose ζ is an n^2-th root of unity. Then ζ^2 is an n-th root of unity.*

   (f) *Let n be a positive integer, and ζ be a complex number. Suppose ζ^n is an n-th root of unity. Then ζ is a $(2n)$-th root of unity.*

3. Consider the statement ($\star$):

   ($\star$) *Suppose A, B, C are sets. Then $A\backslash(C\backslash B) \subset A \cap B$.*

   (a) Write down the negation of the statement ($\star$).

   (b) Dis-prove the statement ($\star$).

4. Dis-prove each of the statements below by giving an appropriate argument. (It may help if you draw Venn diagrams to investigate the respective statements first.)

   (a) *Suppose A, B, C are non-empty sets. Then $B\backslash A \subset (C\backslash A)\backslash(C\backslash B)$.*

   (b) *Suppose A, B, C are non-empty sets. Then $A \cup (B \cap C) \subset (A \cup B) \cap C$.*

   (c)$^\diamond$ *Suppose A, B, C are non-empty sets. Then $B \cap C \subset [A\backslash(B\backslash C)] \cup [B\backslash(C\backslash A)]$.*

   (d)$^\diamond$ *Let A, B, C be sets. Suppose $A \cap B \subset C$. Then $C \subset (A \cap C) \cup (B \cap C)$.*

   (e)$^\diamond$ *Let A, B, C be sets. Suppose $A\backslash B$, $A\backslash C$ are non-empty. Then $A\backslash(B \cap C) \subset (A\backslash B) \cap (A\backslash C)$.*

   (f)$^\diamond$ *Let A, B, C, D be non-empty sets. Suppose $A \subset C$ and $B \subset D$. Further suppose $C \cap D \neq \emptyset$. Then $A \cup B \subset C \cap D$.*

5. Dis-prove the statements below:

   (a) *Let I be an open interval, and $f : I \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable on I, and f is strictly increasing on I. Then $f'(x) > 0$ for any $x \in I$.*

(b) *Let I be an open interval, and $f : I \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable on I, and $f'(x) \geq 0$ for any $x \in I$. Then f is strictly increasing on I.*

(c) *Let I, J be open intervals , and $f : I \cup J \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable at every point of $I \cup J$, and $f'(x) = 0$ for any $x \in I \cup J$. Then f is constant on $I \cup J$.*

6. We introduce/recall the definition for the notion of *non-singularity* for square matrices with real entries:

   Let A be an $(m \times m)$-square matrix with real entries. The matrix A is said to be **non-singular** *if the statement* (NS) *holds:*

   (NS) *For any $\mathbf{x} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.*

   Dis-prove the statements below.

   (a) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are non-singular and $A + B$ is not the zero matrix. Then $A + B$ is non-singular.*

   (b) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are non-singular and $A + B$ is not the zero matrix. Then $A + B$ is singular.*

   (c) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are singular and $A + B$ is not the zero matrix. Then $A + B$ is singular.*

   (d) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are singular and $A + B$ is not the zero matrix. Then $A + B$ is non-singular.*