MATH1050 Proof-writing Exercise 5 (Answers and selected solutions.)

1. (a) **Answer.**

   Let S, T be sets. We say S is a subset of T if the statement (†) holds:

   (†) *For any object x, if $x \in S$ then $x \in T$.*

   (b) **Answer.**

   Let A, B be sets.

   The union of the sets A, B is defined to be the set $\{x \mid x \in A$ or $x \in B\}$.

   (c) **Solution.**

   Let A, B, C, D be sets. Suppose $A \subset C$ and $B \subset D$.

   [We want to deduce $A \cup B \subset C \cup D$. According to definition, this is the same as deducing '*For any object x, if $x \in A \cup B$ then $x \in C \cup D$*'.]

   Pick any object x.

   Suppose $x \in A \cup B$. [We ask whether it is true that $x \in C \cup D$.]

   By the definition of union, $x \in A$ or $x \in B$.

   - (Case 1). Suppose $x \in A$. Then, since $A \subset C$, we have $x \in C$ by the definition of subset relation. Therefore $x \in C$ or $x \in D$. Hence $x \in C \cup D$ by the definition of union.
   - (Case 2). Suppose $x \notin A$. Then $x \in B$. Therefore, since $B \subset D$, we have $x \in D$. Then $x \in C$ or $x \in D$. Hence $x \in C \cup D$.

   Hence, in any case, we have $x \in C \cup D$.

   It follows that $A \cup B \subset C \cup D$.

2. (a) **Answer.**

   Let K, L be sets. We say H is equal to K as sets if both statements ($\star$), ($\star\star$) hold:

   ($\star$) *H is a subset of K.*

   ($\star\star$) *K is a subset of H.*

   (b) ——

3. (a) **Answer.**

   Let A, B be sets.

   The intersection of the sets A, B is defined to be the set $\{x \mid x \in A$ and $x \in B\}$.

   The complement of the set B in the set A is defined to be the set $\{x \mid x \in A$ and $x \notin B\}$.

   (b) i. *Hint.* Apply the method of proof-by-contradiction, starting in this way:

   Let A, B be sets. Suppose $A \subset A \backslash B$.

   Further suppose it were true that $A \cap B \neq \emptyset$.

   Now pick some element of $A \cap B$, and see whether a contradiction can be reached.

   ii. *Hint.* Start in this way:

   Let A, B be sets. Suppose $A \cap B = \emptyset$.

   [We try to deduce 'for any object x, if $x \in A$ then $x \in B$'.]

   Pick any object x. Suppose $x \in A$.

   Then apply the method of proof-by-contradiction to verify '$x \in B$'.

   (c) ——

4. ——

5. ——

6. ——

7. (a) **Answer.**

   Let C be a set. The power set of C is defined to be the set $\{S \mid S$ is a subset of $C\}$.

   (b) *Reminder.* Always remember: $T \in \mathfrak{P}(C)$ iff $T \subset C$.

8. ——