MATH1050 Proof-writing Exercise 4

Advice.

- Some of the questions are concerned with mathematical induction.

  Study the Handout *Argument by mathematical induction*, *Examples on argument by mathematical induction*, *Weierstrass Inequality* (with focus on how mathematical induction is applied) if you have not done so.

- Some of the questions are concerned with concrete examples of uniqueness statements.

  Study how uniqueness statements are formulated in the concrete examples of the discussion on the statement of Division Algorithm in the Handout *Division Algorithm*, and the discussion on the notion of greatest common divisors in the Handout *Euclidean Algorithm*.

  Question (5) of Exercise 4 is also suggestive on what it takes to give a correct argument on uniqueness statements.

- Sometimes you may want to apply the method of proof-by-contradiction within one passage of a proof.

  In this situation, it may be good to start that passage with the words '*we want to verify blah-blah-blah with the method of proof-by-contradiction* '.

  Be reminded that the assumptions used in such a passage of argument (which you hope will lead to a desired contradiction within that passage) must be stated clearly at the beginning of the passage concerned.

1. In this question, take for granted the validity of the statement below:

   ($\sharp$) *For any* $t, u, v \in \mathbb{R}$, *if* $t > 0$, $u > 0$, $v > 0$ *and* $t \neq 1$ *then* $\log_t(uv) = \log_t(u) + \log_t(v)$.

   Apply mathematical induction to prove the statement below:

   ($\sharp\sharp$) *Let* $n \in \mathbb{N}\backslash\{0, 1\}$, *and* r *be a real number greater than* 1. *Suppose* $a_1, a_2, \cdots, a_n$ *are positive real numbers. Then* $\log_r\left(\prod_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} \log_r(a_j)$.

2. (a) Verify the statement ($\flat$):

   ($\flat$) *Let* $a, b \in \mathbb{R}$. *Suppose* $0 \leq a \leq b$. *Then* $\dfrac{a}{1+a} \leq \dfrac{b}{1+b}$.

   (b) i. Prove the statement ($\sharp$) below:

   ($\sharp$) *Let* $a, b \in \mathbb{R}$. *Suppose* a, b *are non-negative. Then* $\dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$.

   ii. Apply mathematical induction to prove the statement ($\sharp\sharp$):

   ($\sharp\sharp$) *Let* $n \in \mathbb{N}\backslash\{0, 1\}$. *Suppose* $x_1, x_2, \cdots, x_n$ *are non-negative real numbers. Then*

   $$\frac{x_1 + x_2 + \cdots + x_n}{1 + x_1 + x_2 + \cdots + x_n} \leq \sum_{j=1}^{n} \frac{x_j}{1 + x_j}.$$

   (c)♣ Applying the results described by ($\flat$), ($\sharp\sharp$), and the Arithmetico-geometrical Inequality, or otherwise, prove the statement ($\natural$).

   ($\natural$) *Let* $n \in \mathbb{N}\backslash\{0\}$. *Suppose* $c_1, c_2, \cdots, c_n$ *are non-negative real numbers. Then*

   $$\frac{c_1 c_2 \cdots c_n}{1 + c_1 c_2 \cdots c_n} \leq \frac{{c_1}^n + {c_2}^n + \cdots + {c_n}^n}{n + {c_1}^n + {c_2}^n + \cdots + {c_n}^n} \leq \sum_{j=1}^{n} \frac{{c_j}^n}{n + {c_j}^n}.$$

3. (a) Prove the statement ($\sharp$):

   ($\sharp$) *Let* a, b, u, v *be positive real numbers. Suppose* $u + v = 1$. *Then* $\sqrt{a^2 u + b^2 v} \geq au + bv$.

   (b)♣ Prove the statement below ($\sharp\sharp$):

   ($\sharp\sharp$) *Let* $n \in \mathbb{N}\backslash\{0, 1\}$. *Suppose* $c_1, c_2, \cdots, c_n, x_1, x_2, \cdots, x_n$ *be positive real numbers.*
   Further suppose $x_1 + x_2 + \cdots + x_n = 1$.
   Then $\sqrt{{c_1}^2 x_1 + {c_2}^2 x_2 + \cdots + {c_n}^2 x_n} \geq c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$.

4. We introduce/recall the definition for the notion of *non-singularity* for square matrices with real entries:

   Let A *be an* $(m \times m)$*-square matrix with real entries. The matrix* A *is said to be* **non-singular** *if the statement* (NS) *holds:*

(NS) *For any* $\mathbf{x} \in \mathbb{R}^m$, *if* $A\mathbf{x} = \mathbf{0}$ *then* $\mathbf{x} = \mathbf{0}$.

(a) Prove the statement below, with reference to the definition:

Let A, B *be* $(m \times m)$*-square matrices with real entries. Suppose* A, B *are non-singular. Then* AB *is non-singular.*

(b) Consider the statement $(*)$:

$(*)$ *The product of any two or more non-singular* $(m \times m)$*-square matrices with real entries is non-singular.*

i. Fill in the blanks below in such a way that the resultant statement $(*')$ is a re-formulation of $(*)$:

$(*')$ *Let* n *be an integer greater than* 1. *Let* ____________ *be* $(m \times m)$*-square matrices with real entries. Suppose* ____________________. *Then* ____________________.

ii. Apply mathematical induction to prove the statement $(*)$.

Remark. You may take the result in part (a) for granted.

5. Consider the statement $(\star)$:

$(\star)$ *Every integer greater than* 1 *is a prime number or a product of at least two prime numbers.*

(a) Fill in the blanks below in such a way that the resultant statement $(\star')$ is a re-formulation of $(\star)$:

$(\star')$ *Let* n ____________________ . *For any integer* m *between* __ *and* n, *the integer* m *is a prime number or* ____________________________ .

(b) Apply mathematical induction to justify the statement $(\star)$.

Remark. You need to recall the definitions for the notions of *divisibility* and *prime numbers.*

6. Prove the statement $(\sharp)$:

$(\sharp)$ *Suppose* S *is a subset of* $\mathbb{R}$. *Then* S *has at most one greatest element.*

7.$^\diamond$ Prove the statement $(\sharp)$.

$(\sharp)$ *Let* ζ *be a complex number. Suppose* ζ *is neither real nor purely imaginary. Then for any complex number* z, *there are at most one real number* a *and at most one real number* b *satisfying* $z = a\zeta + b\overline{\zeta}$.

8. Prove the statement $(\sharp)$. (You do not need, and should not use, any result from *calculus.*)

$(\sharp)$ *Let* p *be a positive real number and* q *be a real number. Suppose* $f(x)$ *be the cubic polynomial given by* $f(x) = x^3 + px + q$. *Then, for each real number* v, *the equation* $f(u) = v$ *with unknown* u *has at most one real solution.*

9.$^\diamond$ *Recall the definitions for the notion of* strict monotonicity *for real-valued functions of one real variable from Assignment 1.*

Prove the statement $(\sharp)$, with direct reference to the definition of *strict monotonicity.*

$(\sharp)$ *Let* I *be an interval in* $\mathbb{R}$, *and* $f, g : I \longrightarrow \mathbb{R}$ *be functions. Suppose* f *is strictly increasing on* I *and* g *is strictly decreasing on* I. *Then there is at most one* $c \in I$ *satisfying* $f(c) = g(c)$.

Remark. Be careful. The functions under consideration in the statement $(\sharp)$ are not assumed to be differentiable. Whatever you have learnt in your *calculus* course which relates (strict) monotonicty with derivatives is unapplicable here.

Further remark. At some stage of the argument, you may find it more convenient to apply the method of proof-by-contradiction.

10. We introduce/recall the notion of *linear independence* (for vectors):

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$. *Suppose* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ *are pairwise distinct.*
Then $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ *are said to be* **linearly independent** *if the statement* (LI) *holds:*

(LI) *For any* $a_1, a_2, \cdots, a_k \in \mathbb{R}$, *if* $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k = \mathbf{0}$ *then* $a_1 = a_2 = \cdots = a_k = 0$.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.

Prove the statement $(\sharp)$, with direct reference to the definition of *linearly independence.* (You do not need, and should not use, any result about *classification of solution sets of systems of linear equations.*)

$(\sharp)$ *For any* $\mathbf{v} \in \mathbb{R}^n$, *there are at most one* $c_1 \in \mathbb{R}$, *at most one* $c_2 \in \mathbb{R}$, ..., *and at most one* $c_k \in \mathbb{R}$ *such that* $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.

11.$^\clubsuit$ Prove the statement $(\sharp)$:

$(\sharp)$ *For any* $z \in \mathbb{C}$, *there is at most one* $w \in \mathbb{C}$ *such that for any* $r > 0$, $|w - z| < r$.

Remark. At some stage of the argument, you may find it more convenient to apply the method of proof-by-contradiction.