MATH1050 Proof-writing Exercise 3 (Answers and selected solutions)

1. (a) **Solution.**

   Let $A = \{\zeta \in \mathbb{C} : |\zeta - i| < 1\}, B = \{\zeta \in \mathbb{C} : |\zeta + i| < 3\}$.

   [*Pictorial roughwork.* Give a sketch of A, B on the Argand plane.
   A is the open disc with centre i and radius 1.
   B is the open disc with centre $-i$ and radius 3.
   The former lies entirely inside the latter.]

   i. We verify $A \subset B$.

      [*Reminder.* This amounts to proving '*for any* $\zeta \in \mathbb{C}$, *if* $\zeta \in A$ *then* $\zeta \in B$'.]

      Pick any $\zeta \in \mathbb{C}$. Suppose $\zeta \in A$.
      We have $|\zeta - i| < 1$ (by the definition of A).
      By the Triangle Inequality, we have $|\zeta + i| = |\zeta - i + 2i| \leq |\zeta - i| + |2i| = |\zeta - i| + 2 < 1 + 2 = 3$.
      Then $|\zeta + i| < 3$. Therefore, we have $\zeta \in B$ (by the definition of B).
      It follows that $A \subset B$.

   ii. We verify $B \not\subset A$.

      [*Reminder.* This amounts to proving '*there exists some* $\zeta_0 \in \mathbb{C}$ *such that* $\zeta_0 \in B$ *and* $\zeta_0 \notin A$'.]

      Take $\zeta_0 = 0$. By definition, $\zeta_0 \in \mathbb{C}$.
      Note that $|\zeta_0 + i| = |0 + i| = 1 < 3$.
      Then $\zeta_0 \in B$.
      We verify that $\zeta_0 \notin A$:

      - We have $|\zeta_0 - i| = |0 - i| = 1 \geq 1$.
        Then $\zeta_0 \notin A$.

      It follows that $B \not\subset A$.

   (b) *Hint.*

   D is the closed disc with centre 0 and radius 5.

   E is the closed elliptical region with centre 0, foci at $4, -4$, vertices at $5, -5$ and covertices $3i, -3i$. (Its boundary is given by the equation $\dfrac{(\mathsf{Re}(z))^2}{25} + \dfrac{(\mathsf{Im}(z))^2}{9} = 1$ with complex unknown z.)

   Answer.

   i. $D \not\subset E$.
      Hint for the argument.
      $5i$ belongs to D and does not belong to E.

   ii. $E \subset D$.
      Hint for the argument.
      The key step is to make use of the inequality '$|2z| \leq |z - 4| + |z + 4|$' which holds for any arbitrary complex number z.

   (c) *Hint.*

   E is the closed elliptical region with centre 0, foci at $4, -4$, vertices at $5, -5$ and covertices $3i, -3i$.

   F is the closed disc with centre 0 and radius 3.

   Answer.

   i. $E \not\subset F$.
      Hint for the argument.
      4 belongs to E and does not belong to F.

   ii. $F \subset E$.
      Hint for the argument.
      Make use of the relation '$(|z - 4| + |z + 4|)^2 = 2|z|^2 + 32 + 2|z^2 - 16|$' which holds for any arbitrary complex number z.
      Also make use of the inequality '$|z^2 - 16| \leq |z^2| + 16$' which holds for any arbitrary complex number z.

2. **Answer.**

(a) i. $A \subset B$.

ii. $B \not\subset A$.

Hint on the argument.

$-\frac{1}{9}$ belongs to B and does not belong to A.

(b) i. $C \subset D$.

ii. $D \not\subset C$.

Hint on the argument.

2 belongs to D and does not belong to C.

3. **Answer.**

(a) $B \subset A$.

(b) $A \not\subset B$.

Hint on the argument.

12 belongs to A and does not belong to B.

4. **Answer.**

(a) $A \subset B$.

(b) $B \not\subset A$.

Hint on the argument.

γ^2 belongs to B and does not belong to A.

(In fact, for any $a, b, c \in \mathbb{Q}$, if $c \neq 0$ then $a + b\gamma + c\gamma^2$) belongs to B and does not belong to A.

The key idea in the argument for '$\gamma^2 \notin A$' is described below:

> If $\gamma^2 \in A$, then, with the help of the relation $\gamma^3 = 2$, we would obtain a relation of the form $p\gamma = q$, in which p, q are some appropriate rational numbers. This in turn leads to a contradiction.

5. ——