MATH1050 Proof-writing Exercise 3

**Advice.**

- All the questions are concerned with concrete examples of subset relations.

  Study the Handout *Examples of proofs concerned with 'subset relations'*.

- In some questions a careful handling of '*there exists*' is needed. In some questions, you need to recall how to handle *divisibility* for integers, or *rationals and irrationals*, or *modulus* for complex numbers.

  Study the Handouts *Basic results on divisibility*, *Rationals and irrationals*, *Basic algebraic results on complex numbers 'beyond school mathematics'*, where appropriate.

- Besides the handouts mentioned above, Question (15) of Assignment 3 may also be relevant.

- When giving an argument, remember to adhere to definition, always. And don't forget to make the argument self-contained: the assumptions needed in the argument should be stated, (likely) at the the top.

- Sometimes you may want to apply the method of proof-by-contradiction within one passage of a proof.

  In this situation, it may be good to start that passage with the words '*we want to verify blah-blah-blah with the method of proof-by-contradiction*'.

  Be reminded that the assumptions used in such a passage of argument (which you hope will lead to a desired contradiction within that passage) must be stated clearly at the beginning of the passage concerned.

1. In this question, make good use of the **Triangle Inequality** where appropriate:

   - *Suppose $z, w \in \mathbb{C}$. Then $|z+w| \leq |z| + |w|$. Moreover, equality holds iff one of $z, w$ is a non-negative real multiple of the other.*

   Also, try not to 'break' a complex number into its real and imaginary parts.

   (a) Let $A = \{\zeta \in \mathbb{C} : |\zeta - i| < 1\}, B = \{\zeta \in \mathbb{C} : |\zeta + i| < 3\}$.

      i. Is it true that $A \subset B$? Justify your answer.

      ii. Is it true that $B \subset A$? Justify your answer.

   (b) Let $D = \{\zeta \in \mathbb{C} : |\zeta| \leq 5\}$, $E = \{\zeta \in \mathbb{C} : |\zeta - 4| + |\zeta + 4| \leq 10\}$.

      i.$^\diamondsuit$ Is it true that $D \subset E$? Justify your answer.

      ii.$^\diamondsuit$ Is it true that $E \subset D$? Justify your answer.

   (c) Let $E = \{\zeta \in \mathbb{C} : |\zeta - 4| + |\zeta + 4| \leq 10\}$, $F = \{\zeta \in \mathbb{C} : |\zeta| \leq 3\}$.

      i.$^\diamondsuit$ Is it true that $E \subset F$? Justify your answer.

      ii.$^\clubsuit$ Is it true that $F \subset E$? Justify your answer.

2. (a) Let $A = \{x \mid 3x = 8n + 1 \text{ for some } n \in \mathbb{Z}\}$, $B = \{x \mid 9x = 4n - 1 \text{ for some } n \in \mathbb{Z}\}$.

      i. Is it true that $A \subset B$? Justify your answer.

      ii. Is it true that $B \subset A$? Justify your answer.

   (b) Let $C = \{x \mid x = 12m + 18n \text{ for some } m, n \in \mathbb{Z}\}$, $D = \{x \mid x = 6m + 8n \text{ for some } m, n \in \mathbb{Z}\}$.

      i. Is it true that $C \subset D$? Justify your answer.

      ii. Is it true that $D \subset C$? Justify your answer.

3. Let $A = \{x \mid x = 3r^2 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \mid x = 3r^{10} \text{ for some } r \in \mathbb{Q}\}$.

   (a) Is it true that $B \subset A$? Justify your answer.

   (b) Is it true that $A \subset B$? Justify your answer.

   **Remark.** In the argument you may take for granted the validity of the statement below, where appropriate and necessary:

   - *Suppose $n$ is an integer greater than 1, and $p$ is a positive prime number. Then the number $\sqrt[n]{p}$ is irrational.*

4. Write $\gamma = \sqrt[3]{2}$. Take for granted that $\gamma$ is irrational.

Let $A = \{x \mid x = a + b\gamma \text{ for some } a, b \in \mathbb{Q}\}$, $B = \{x \mid x = a + b\gamma + c\gamma^2 \text{ for some } a, b, c \in \mathbb{Q}\}$.

(a) Is it true that $A \subset B$? Justify your answer.

(b)♣ Is it true that $B \subset A$? Justify your answer.

5.♣ We introduce/recall the definition for the notions of *null space* and *column space* for matrices (with real entries):

*Let $A$ be an $(m \times n)$-matrix with real entries.*

* *The null space of $A$ is defined to be the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}_m\}$.*
*It is denoted by $\mathcal{N}(A)$.*

* *The column space of $A$ is defined to be the set* $\left\{\mathbf{y} \in \mathbb{R}^m : \begin{array}{l} \text{There exists some } \mathbf{x} \in \mathbb{R}^n \\ \text{such that } \mathbf{y} = A\mathbf{x} \end{array}\right\}$.

*It is denoted by $\mathcal{C}(A)$.*

(a) Prove the statements below, with reference to the definition of *column space*.

i. *Suppose $B$ is an $(n \times n)$-square matrix with real entries. Then $\mathcal{C}(B^2) \subset \mathcal{C}(B)$.*

ii. *Let $D$ be an $(m \times n)$-matrix with real entries, and $E, F$ be $(n \times p)$-matrices with real entries. Suppose $\mathcal{C}(E) \subset \mathcal{C}(F)$. Then $\mathcal{C}(DE) \subset \mathcal{C}(DF)$.*

(b) Prove the statement below, with reference to the respective definitions of *null space* and *column space*.

- *Suppose $G$ is an $(m \times n) -$ matrix with real entries, and $H$ be an $(n \times p)$-matrix with real entries. Then $GH = \mathcal{O}_{m \times p}$ iff $\mathcal{C}(H) \subset \mathcal{N}(G)$.*